BAYESIAN MODELS FOR SEQUENTIAL BIDDING AND RELATED THEORETICAL TOPICS

by

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DECLARATION

Unless otherwise stated all the work in this thesis is my own. Chapter 6 was developed and adapted through discussion with my supervisor Dr. Jim Smith.

SUMMARY

The problems faced by a company in determining its bids on a sequence of contracts put out to tender has been studied extensively in past literature. This thesis concentrates on formalising and developing these existing models as well as presenting new models for both the competing bidders and the tendering organisation.

Development of these models involves reviewing and extending theoretical results in the areas of time series analysis, multi–armed bandits, and finitely additive probability distributions.

1. INTRODUCTION

1.1 The Sequential Bidding Problem.

Organisations such as public authorities and large companies are often required to employ outside contractors to carry out work for them. The usual method of awarding such a contract is to put it out to *tender*, as follows. The organisation will invite a small number of companies, from a larger pool of companies, to submit a bid, or estimate, of their charge for the work. It is common for the company submitting the lowest bid to be automatically awarded the contract, especially when the work is of a moderate cost. If the same organisation is sequentially tendering similar contracts to the same pool of companies we shall refer to the problems faced by the tendering organisation (tenderer) and bidding companies (bidders) as *The Sequential Bidding Problem*. The aim of this thesis is to present mathematical models for both the tenderer and the bidders to assist in their decision making process in a sequential bidding environment.

The tenderer's problem is, for each contract, to select a subset of the pool of companies who, on the one hand, he believes will between them quote a low bid, and on the other hand will provide him with useful information as to which companies to invite to bid in the future. When formulated mathematically this leads to a stochastic control problem which is a generalisation of the well known multi–armed bandit problem.

The bidder's problem is to select a bid that is large enough to ensure a reasonable profit, but small enough to have a significant chance of winning. He might also take into account the fact that consistently high bidding may alienate the tenderer and result in him not being invited to bid on future contracts.

Before discussing the problem further, and outlining the aims of this thesis, it is worth emphasising the appropriateness of the Bayesian approach for modelling this problem. Most practitioners would accept that the role of a mathematical sequential decision model is to provide and update basic parameters that the models operator should then use, in conjunction with any subjective information he may have, to formulate a decison at each time period. A Bayesian model has the attractive feature that it permits, and indeed encourages, the incorporation of subjective information at all time periods. This is particularly useful in a sequential bidding environment where subjective information is likely to be readily available. For example one may learn that a particular bidder is likely to be behaving differently for the next contract or two, for, say, financial reasons.

We now consider the possible approaches to modelling the problem.

An ambitious way to model the whole problem is to use a game theoretic model with the tenderer and the individual bidders as players, i.e. a model where it is assumed *all* players are behaving rationally. An appropriate model may be based on the ideas of Harsanyi (1967,1968a,b). The basic principles underlying such a model are briefly outlined in chapter 8. It should be emphasised that many player games of this type are immensely complex, and so, in order to make progress, we must look to simplify the problem somewhat.

The first step in this simplification is to look at the problems facing the tenderer and bidders separately. As mentioned above, the tenderers problem can be formulated as a stochastic control problem – this is the subject of chapter 7. With the problem faced by an individual bidder we have two choices of approach. The problem can be modelled in a game theoretic way with just the competing bidders as players, or alternatively we can look at the problem faced by an individual bidder in a decision theoretic framework. The decision theoretic approach means we assume our competitors are behaving predictably in some sense, for example drawing their bids randomly from some fixed distribution. This approach is the subject of chapters 4, 5 and 6.

Game theoretic models with just the bidders as players have received some attention in the literature, stemming from the papers of Griesmer & Shubik (1967a,b,c). Other contributions have come from Smith & Case (1975), Engelbrecht–Wiggins (1980), Holt (1980) and King & Mercer (1988). However the vast majority of the bidding literature is devoted to the decision theoretic approach to the bidders problem, mostly revolving around the papers of Friedman (1956) and Gates (1967). Chapter 3 provides a review of this material.

Summarising, this thesis does not consider any game theoretic models. In the main we concentrate on formalising and extending existing decision theoretic models for the bidder's and tenderer's problems as well as presenting new models and examining related topics. We shall thus now develop the basic notation and assumptions for the problem.

The first point to note is that in order to eliminate the effects of the different costs of different contracts it is common to work in terms of a companies *markup*, m, rather than their bid, b. If we consider ourselves as assisting an individual bidder in determining their bids, then we define our markup as $m = b/c$, the ratio of our bid to our point estimate of the cost to us of fulfilling the contract, c. We shall define the other bidders markups as the ratio of *their* bid to *our* point estimate of the cost of the contract, c, we thus might refer to this as our competitors *apparent* markup. It is worth noting that this definition of markup is equivalent to defining a competitors markup relative to their own cost and then assuming that ourselves and all our competitors are using the same point estimate of the cost of the contract.

Our bidder's decision problem is now to select a markup. Our actual bid will then be given by this markup times our estimate of the cost to us of fulfilling the contract. Of course the true cost will be a random variable, C, since at this stage the final cost will be unknown to us.

The simplest criteria on which to base the selection of our markup is to maximise immediate expected profit $(m-1)\mathbf{E}\left[Cp(m)\right]$ where $p(m)$ is the probability that we win the contract with markup m. This probability is clearly uncertain to us and so, as Bayesians, we must treat it as a random variable, indexed by m. We now make the assumption that the random variables C and $p(m)$ are independent for all values of m. In practice this means that we are assuming our competitors will not change their procedure for choosing a markup for contracts of different costs. If the sequence of contracts have similar costs this seems a very reasonable assumption. With this assumption the expected profit can now be written as $c(m-1)\bar{p}(m)$ where $c = \mathbf{E}[C]$ and $\bar{p}(m) = \mathbf{E}[p(m)]$. Thus the problem reduces to computing an estimate of $\bar{p}(m)$– this is the starting point for most of the existing models discussed in chapter 3.

As mentioned above, this thesis mainly concentrates on *mathematical* models. It is however intended that the models presented are flexible enough to allow some of the many practical problems encountered in a real bidding environment to be incorporated. In some cases this is actually done, and in others it is indicated how it may be done. We thus close this introduction to the sequential bidding problem by mentioning a few of these practical considerations, the main references here are Ward & Chapman (1988) and King & Mercer (1985).

(i)*More realistic utilities* : In the discussion above the criteria used to choose a markup was simply maximisation of expected profit. In reality a bidder may wish to consider other properties of a contract, for example the prestige associated with winning a given contract. Or he may be happy to take a smaller markup for a very costly contract. One way to overcome this is to define a utility taking into account the relative importance of expected profit and these other features. Our markup would then be chosen to maximise this utility. An example of such a utility is given in §4.3. If we wish to model our competitors as also reacting to factors such as prestige and cost we must look to build a model that can handle these factors as regressors in the estimation of $\bar{p}(m)$. A model of this type is discussed in §4.5.

(ii)*Non-price factors* : A similar consideration to the above is that of non-price factors i.e. factors, other than profit, that we might wish to consider when choosing our markup. An example of this is our available resources at the time of each contract. Clearly if we win a contract and do not have the available resources to do the work we will incur a cost in employing further staff or buying more equipment in order to fulfill the contract. So the event of us winning the contract at time t, which will clearly affect our resources at time $t+1$, will tend to make us bid less competitively at time $t+1$ as we will need a larger profit to cover any extra costs incurred as a result of having less resources available. Mathematical formulation of this leads to a stochastic control problem – a brief review of the literature on this is given in chapter 3.

(iii)*Seasonal effects* : Clearly, in many bidding environments, such as building works, factors such as tempereature and number of daylight hours are going to affect the number of contracts available and thus the level of competition for these contracts. These seasonal effects can easily be incorporated into the specification of $\bar{p}(m)$. This is discussed in sections 4.3 and 4.5.

(iv)*Cost estimation* : One of the major problems with attempting to model a competitors behaviour is that the variability in their cost estimate can sometimes swamp subtle changes in their markup policy. Some authors have gone as far as concerning themselves more with estimating the cost that a competitor is likely to come up with than with their markup, see for example Naert & Weverbergh (1978). In chapter 5 we present a model which attempts to overcome this by seperating the variability in cost, and that in markup.

(v)*'one off' information about competitors* : It is common for a bidder to obtain information about competitors other than that found out simply from their bidding behaviour. For example it may be discovered that a competitor is in financial difficulty, or has a problem with resources. Information of this kind is clearly important in determining a bid as it will affect a competitors behaviour. It is thus desirable that models have the ability to incorporate this type of information. This is discussed in §4.5 in relation to the models of chapter 4.

(vi) *The tenderer's non-price criteria* : Clearly if it is believed that the tenderer is not neccessarily going to award the contract to the lowest bidder, this will have a significant effect on the estimation of $p(m)$. Ward & Chapman (1988) discuss this point in detail. However for contracts of a reasonable size and importance it is very likely that a tenderer would have sufficient knowledge of potential bidders to only invite companies to bid whom he was sure would be capable of fulfilling the contract satisfactorily. If we assume this to be the case then there is no reason to suppose that the tenderer would not award the contract to the lowest bidder. We shall assume this to be the case throughout the thesis.

1.2 Schemata.

This thesis essentially divides into four parts. Chapters 2 and 3 provide background and review material. Chapters 4 and 5 present two types of decision theoretic models for the bidders problem. Chapter 6 then examines the relationship between these two types of model and develops some interesting theoretical asides. Finally chapter 7 looks at the decision theoretic approach to the tenderers problem.

One of the main tools in any sequential decision model is likely to be time series analysis and accordingly chapter 2 is devoted to reviewing standard Bayesian time series models which are then used in chapters 4,5 and 7. As mentioned above, chapter 3 provides a review of existing decision theoretic models for the bidders problem.

The first part of chapter 4, sections 4.1–4.3, formalises one of these decision theoretic models, the Gates model, and develops it into a sequential model. Section 4.4 develops a multivariate analogue to one of the time series models mentioned in chapter 2 and §4.5 then applies this to the model described earlier in chapter 4 to provide a powerful sequential version of the Gates model. Similarly chapter 5 produces a sequential version of another model discussed in chapter 3, the Friedman model.

Chapter 6 examines the relationship between the Gates and Friedman models and as a result of this defines and characterises a new class of finitely additive distributions.

The tenderers decision problem is formulated in chapter 7 and shown to be a generalisation of the standard multi-armed bandit problem (MABP). The MABP is then reviewed and an attempt to extend the solution ideas for the MABP to the more general tenderers problem is given.

Chapter 8 then addresses some issues that there has not been time to develop in the main part of the thesis.

Finally it is worth noting that in a thesis such as this which addresses the same basic problem from a number of different viewpoints it is essential to get clear in ones mind exactly which problem is being tackled in each chapter. Thus at the begining of chapters 4,5,6 and 7 a sort of mini introduction is provided to clarify the assumptions used, the problem addressed and the likely applicability of the model described in that chapter.

2. A REVIEW OF BAYESIAN TIME SERIES MODELS

2.1 The Bayesian Approach to Time Series Analysis.

The classical ARIMA time series models of Box & Jenkins (1970) work on the idea that a series can be modelled by first detecting and removing non-stationary components, such as trend and seasonality, and modelling the resulting stationary series as a sum of random variables, which they later assume to be normal. Two practical criticisms can be levelled at these models. Firstly, any attempt to move away from the normality of the components in the model for the stationary series will lead to problems of tractability in the model fitting. And secondly, the attempt to remove all the features of the series before any modelling takes place does not allow any room for heuristic reasoning in the model building process, which would be very desirable if the modeller had a good knowledge of a particular series. A further more general, but related, criticism is that the models are too rigid and do not easily allow an operator to incorporate any extra information he may receive as the model is running.

These problems led practicioners to develop an alternative class of Bayesian time series models. Bather (1965) proposed a class of non-normal state space models which he showed to be fully tractable. A very flexible normal state space model, the *Dynamic Linear Model* (DLM), was introduced by Harrison & Stevens (1976) as follows. An observed series $\{Y_t\}$ is related via a linear model to an underlying markov series $\{\boldsymbol{\theta}_t\}$ which has some heuristic meaning and is consequently known as the state, or level, of the series. This is summarised in the equations below,

$$\text{Observation equation :} \quad Y_t = \mathbf{F}_t\boldsymbol{\theta}_t + V_t \qquad V_t \sim N(0, V) \qquad (2.1.1)$$

$$\text{System equation :} \quad \boldsymbol{\theta}_t = \mathbf{G}_t\boldsymbol{\theta}_{t-1} + \mathbf{W}_t \qquad \mathbf{W}_t \sim N(0, \mathbf{W}) \qquad (2.1.2)$$

the error components V_t and $\mathbf{W}_t$ are independent, with the values of V and $\mathbf{W}$ being prespecified. The vector $\mathbf{F}_t$ is a known vector of regressors as in the ordinary static linear model, and the matrix $\mathbf{G}_t$ is a known state transition matrix which can be used to model factors such as growth trend and seasonality. The analysis of the model proceeds as follows.

Let D_t denote all the data up to and including time t, and take the prior distribution

$$\boldsymbol{\theta}_{t-1} \sim N(\mathbf{m}_{t-1}, \mathbf{C}_{t-1})$$

then (2.1.1) and (2.1.2) give

$$\boldsymbol{\theta}_t \mid D_{t-1} \sim N(\mathbf{a}_t, \mathbf{R}_t)$$

$$\boldsymbol{\theta}_t \mid D_t \sim N(\mathbf{m}_t, \mathbf{C}_t)$$

where the updated values $\mathbf{m}_t$, $\mathbf{C}_t$ are given by the *Kalman filter* equations, Kalman (1963), as follows,

$$\begin{aligned}
\mathbf{a}_t &= \mathbf{G}_t \mathbf{m}_{t-1} \\
\mathbf{R}_t &= \mathbf{G}_t \mathbf{C}_{t-1} \mathbf{G}_t^T + \mathbf{W} \\
\mathbf{m}_t &= \mathbf{a}_t + \mathbf{s}_t e_t / (\mathbf{F}_t^T \mathbf{R}_t \mathbf{F}_t + V) \\
\mathbf{C}_t &= \mathbf{R}_t - \mathbf{s}_t \mathbf{s}_t^T / (\mathbf{F}_t^T \mathbf{R}_t \mathbf{F}_t + V)
\end{aligned} \tag{2.1.3}$$

with $\mathbf{s}_t = \mathbf{R}_t \mathbf{F}_t$ and $e_t = Y_t - \mathbf{F}_t^T \mathbf{a}_t$. Kalman originally derived these equations for updating the first two cumulants of $\boldsymbol{\theta}_t$ without distributional assumptions. Harrison & Stevens showed that they were also the appropriate equations with the state vector and observations being normal. Harrison & Stevens then go on to provide full forecast distributions for $\{Y_{t+k} \mid D_{t-1}\}$.

An important special case is when $\mathbf{F}_t = \mathbf{G}_t = \mathbf{I}$, the identity matrix, for all t. This is called the *steady model* since the evolution on $\boldsymbol{\theta}_t$ given by (2.1.2) results in its mean remaining unchanged but its variance growing at a constant rate. Smith & Miller (1986) develop a non-normal steady model, with a multiplicative system equation, similar to one given by Bather (1965). They then apply this model to the prediction of records.

Ameen & Harrison (1985) discussed the difficulty of specifying a value for $\mathbf{W}$. First they noted that the updating from $\mathbf{C}_{t-1} \to \mathbf{R}_t$ in (2.1.3) simply has the effect of increasing the variance components of $\mathbf{C}_{t-1}$. They then showed that for the normal DLM the updating given by

$$\mathbf{R}_t = \mathbf{B}_t \mathbf{G}_t \mathbf{C}_{t-1} \mathbf{G}_t^T \mathbf{B}_t$$

where $\mathbf{B}_t$ is a diagonal *discount* matrix whose elements are all greater than 1, is formally identical to the updating from $\mathbf{C}_{t-1} \to \mathbf{R}_t$ in (2.1.3). Smith (1988) has taken these ideas further and shown that when $\mathbf{G}_t = \mathbf{I}$ this updating, which he refers to as a *spread steady model*, is a sound method for representing dissolution of information in non-normal models. This will prove very useful in §2.3.

The DLM overcomes many of the limitations of of the ARIMA models. The series is modelled directly which positively encourages input from an informed modeller, both in the model specifications and via intervention when the model is running. Extensions away from the normality assumptions are the subject of the following two sections.

The next section reviews an evolution on the state θ_t that is simply applicable to all distributions and is in all important respects the same as the steady model described above. In §2.3 we review the *Dynamic Generalised Linear Model* which attempts to generalise the full DLM to non-normal distributions

2.2 The Power Steady Model.

Consider first the steady DLM where θ_t is a univariate parameter. The observation equation (2.1.1) is clearly equivalent to saying

$$Y_t \mid \theta_t \sim N(\theta_t, V)$$

which can easily be generalised to non-normal likelihoods for $Y_t|\theta_t$. The crux of the model however is the evolution of the parameter θ_t, that is

$$\theta_t \mid D_t \rightarrow \theta_{t+1} \mid D_t \tag{2.2.1}$$

In the steady DLM this is given by the additive equation (2.1.2) which, with a normal distribution on θ_t, presents no problems as normality is preserved under addition. However if θ_t is non-normal, an additive equation such as (2.1.2) presents problems of tractability in computing the distribution of $\theta_{t+1} \mid D_t$.

Smith (1979) took the view that rather than state the form of the evolution (2.2.1) and then examine its consequences, one should define the properties one wants an evolution to have and then look for a transformation on the density of $\theta_t|D_t$ that has these properties. Smith chose to define his desired properties on a decision space associated with the parameter θ_t as follows,

(1) The Bayes decision, with respect to any step loss function, on the random variable $\theta_{t+1}|D_t$ should be the same as the Bayes decision on the random variable $\theta_t|D_t$

(2) If $\theta_t|D_t$ has a unimodal density, then the expected loss, with a step loss function, of a Bayes decision on $\theta_{t+1}|D_t$ should be at least as large as the expected loss of the same Bayes decision on $\theta_t|D_t$

These properties effectively define what Smith means by a steady evolution on θ_t. Smith then proves that the only transformation on the density of $\theta_t | D_t$ to satisfy these properties, as well as some regularity conditions, is of the form

$$f_{t+1}(\theta_{t+1} \mid D_t) \propto [f_t(\theta_t \mid D_t)]^{k_t} \qquad 0 < k_t \leq 1 \text{ for all t} \qquad (2.2.1)$$

where f_{t+1} is the p.d.f. of $\theta_{t+1}|D_t$ and f_t is the p.d.f. of $\theta_t|D_t$. Smith called the process θ_t undergoing the evolution (2.2.1) a ***power steady model***, and if $k_t = k$ for all t he called it a ***simple power steady model***. One feature of the model is that, unlike a model using a well defined system equation, it does not provide the joint distribution across $\{\boldsymbol{\theta}_{t+1}, \boldsymbol{\theta}_t \mid D_t\}$. At first sight this may appear to be a deficiency of the model. However Smith & Miller (1986) and Smith (1988,89a) argue that this is not so, pointing out that the crux is that the model provides well defined forecast distributions across observables.

In practical terms the model is very useful as the evolution (2.2.1) is easily applicable to non-normal distributions on θ_t. It is also very flexible in that the evolution can be applied to transformations of the parameterisation of θ_t, allowing the evolution to be applied to ***natural*** parameterisations. For example in §4.1 the evolution (2.2.1) is applied to a multivariate logistic transform of a set of probabilities.

An interesting property is that, unlike the steady DLM, the evolution (2.2.1) does not neccesarily imply that $\mathbf{E}[\theta_{t+1} \mid D_t] = \mathbf{E}[\theta_t \mid D_t]$, it is however always true that $mode\,[\theta_{t+1} \mid D_t] = mode\,[\theta_t \mid D_t]$.

Smith (1981) develops these ideas to a multivariate parameter $\boldsymbol{\theta}_t$. The natural extension is to take an evolution of the form

$$f_{t+1}(\boldsymbol{\theta}_{t+1} \mid D_t) \propto [f_t(\boldsymbol{\theta}_t \mid D_t)]^k \qquad 0 < k \leq 1$$

which is perfectly valid but has the slightly restrictive implication that $\theta_t^{(i)} \mid \boldsymbol{\theta}_t \setminus \{\theta_t^{(i)}\}$ undergoes a simple power steady evolution with the same value of k for each i, $\theta_t^{(i)}$ being the i^{th} element of $\boldsymbol{\theta}_t$.

For situations where it is believed that this symmetrical development on the elements of $\boldsymbol{\theta}_t$ is inappropriate, Smith proposed the ***stacked steady model*** as follows. First reorder the elements of $\boldsymbol{\theta}_t$, $(\theta_t^{(1)}, \ldots, \theta_t^{(n)}) \rightarrow (\theta_t^1, \ldots, \theta_t^n)$ say, such that we believe information about θ_t^{i+1} is lost quicker than information about θ_t^i for $i = 1, \dots, n-1$. The stacked steady model is now defined by letting $\theta_t^i \mid \theta_t^{i+1}, \dots, \theta_t^n$ develop with a simple power steady model with discount factor k_i for $i = 1, \dots, n$, where $0 < k_i < k_{i+1} \leq 1$, $i = 1, \dots, n-1$

2.3 The Dynamic Generalised Linear Model.

In this section we review the Dynamic Generalised Linear Model (DGLM) as proposed by West, Harrison & Migon (1985). The DGLM attempts to generalise the full DLM, as defined by (2.1.1) and (2.1.2), to non–normal distributions on the observed series and the state. The price that has to be paid to make this generalisation work is that the state $\boldsymbol{\theta}_t$ is modelled as a second order process, i.e. only information about its first two cumulants is maintained, and no distributional assumptions are made. The model does however provide a full distribution across the observables $\{Y_t\}$. The model is defined as follows.

In place of the observation equation (2.1.1) observations are assumed to come from an exponential family likelihood as follows,

$$p(Y_t \mid \eta_t, \phi) \propto exp\{\phi(\eta_t Y_t - a(\eta_t))\} \tag{2.3.1}$$

where η_t is the ***natural parameter*** for Y_t. The system equation is as before i.e.

$$\boldsymbol{\theta}_t = \mathbf{G}_t \boldsymbol{\theta}_{t-1} + \mathbf{W}_t \tag{2.3.2}$$

but now the distribution of $\mathbf{W}_t$ is unspecified. The parameters η_t and $\boldsymbol{\theta}_t$ are connected via the ***guide relationship***

$$g(\eta_t) = \mathbf{F}_t^T \boldsymbol{\theta}_t \tag{2.3.3}$$

where $g(.)$ is a known function, quite possibly $g(x) = x$, and $\mathbf{F}_t$ is a known vector as in the DLM.

The priors on η_{t-1} and $\boldsymbol{\theta}_{t-1}$ are as follows,

$$\eta_{t-1} \mid D_{t-1} \sim CP[\alpha_{t-1}, \beta_{t-1}] \tag{2.3.4}$$

$$\boldsymbol{\theta}_{t-1} \mid D_{t-1} \sim [\mathbf{m}_{t-1}, \mathbf{C}_{t-1}] \tag{2.3.5}$$

where $CP[.,.]$ denotes the conjugate prior for the likelihood (2.3.1) which has p.d.f.

$$p(\eta) \propto exp\{\alpha\eta - \beta a(\eta)\} \tag{2.3.6}$$

Equation (2.3.5) does not specify a full distribution for $\boldsymbol{\theta}_{t-1} \mid D_{t-1}$ but simply says it has cumulants given by

$$\mathbf{E}[\boldsymbol{\theta}_{t-1} \mid D_{t-1}] = \mathbf{m}_{t-1}$$
$$Var[\boldsymbol{\theta}_{t-1} \mid D_{t-1}] = \mathbf{C}_{t-1}$$

The development (2.3.2) on $\boldsymbol{\theta}_{t-1}$ gives

$$\eta_t \mid D_{t-1} \sim CP\,[\,\alpha'_t, \beta'_t\,] \tag{2.3.7}$$

$$\boldsymbol{\theta}_t \mid D_{t-1} \sim [\mathbf{a}_t, \mathbf{R}_t] \tag{2.3.8}$$

where

$$\mathbf{a}_t = \mathbf{G}_t \mathbf{m}_{t-1} \tag{2.3.9}$$
$$R_t = \mathbf{B}_t \mathbf{G}_t \mathbf{C}_{t-1} \mathbf{G}_t^T \mathbf{B}_t$$

Note that the updating from $\mathbf{C}_{t-1} \rightarrow \mathbf{R}_t$ is achieved by using a discount matrix $\mathbf{B}_t$, which, as noted in §2.1, is a perfectly sound method for representing dissolution of information about $\boldsymbol{\theta}_t$ in this non-normal environment.

In order to obtain the values of α'_t, β'_t one can use the guide relation (2.3.3) as follows. Define

$$f_t = \mathbf{F}_t^T \mathbf{a}_t \tag{2.3.11}$$
$$q_t = \mathbf{F}_t^T \mathbf{R}_t \mathbf{F}_t$$

then from (2.3.7) compute $\mathbf{E}[g(\eta_t) \mid D_{t-1}]$ and $Var[g(\eta_t) \mid D_{t-1}]$ as functions of α'_t, β'_t. It is then clear, that by taking the mean and variance of each side of (2.3.3) and equating them, that the following equations can be used to solve for α'_t, β'_t as functions of $\mathbf{a}_t, \mathbf{R}_t, f_t$ and q_t.

$$\mathbf{E}[g(\eta_t) \mid D_{t-1}] = f_t \tag{2.3.12}$$
$$Var[g(\eta_t) \mid D_{t-1}] = q_t$$

Consider now updating the parameters $\alpha'_t, \beta'_t, \mathbf{a}_t$ and $\mathbf{R}_t$ in the light of an observation Y_t, hence obtaining the updates for the distributions $\eta_t \mid D_t$ and $\boldsymbol{\theta}_t \mid D_t$.

Bayes theorem and equations (2.3.1) and (2.3.6) immediately give

$$\eta_t \mid D_t \sim CP\,[\,\alpha_t' + \phi Y_t\,, \beta_t' + \phi\,] \tag{2.3.13}$$

There are of course problems in updating $\mathbf{m}_{t-1}$ and $\mathbf{C}_{t-1}$ as we do not know the distribution of $\boldsymbol{\theta}_t \mid D_{t-1}$, only its cumulants. West et. al. overcame this as follows. First note the identities

$$\begin{aligned} \mathbf{m}_t &= \mathbf{E}[\,\mathbf{E}[\boldsymbol{\theta}_t \mid \eta_t\,, D_t]\,] \\ \underline{\mathbf{C}}_t &= Var[\mathbf{E}[\boldsymbol{\theta}_t \mid \eta_t\,, D_t]\,] + \mathbf{E}[Var[\boldsymbol{\theta}_t \mid \eta_t\,, D_t]\,] \end{aligned} \tag{2.3.14}$$

where in all cases the first operator is with respect to η_t. These are useful since η_t is sufficent for $\boldsymbol{\theta}_t$ (see West et. al. for details) and hence the distributions of $\boldsymbol{\theta}_t \mid \eta_t, D_t$ and $\boldsymbol{\theta}_t \mid \eta_t\,, D_{t-1}$ are the same. Thus if one can compute the cumulants $\mathbf{E}[\boldsymbol{\theta}_t \mid \eta_t\,, D_{t-1}]$ and $Var[\boldsymbol{\theta}_t \mid \eta_t\,, D_{t-1}]$, which are not directly dependent on Y_t, these can be fed into (2.3.14) to obtain the required updating. The computation of these cumulants is not easy as the only information one has are the cumulants of $g(\eta_t), \boldsymbol{\theta}_t \mid D_{t-1}$ i.e.

$$\begin{pmatrix} g(\eta_t) \\ \boldsymbol{\theta}_t \end{pmatrix} \mid D_{t-1} \sim \left[\begin{pmatrix} f_t \\ \mathbf{a}_t \end{pmatrix}, \begin{pmatrix} q_t & \mathbf{s}_t^T \\ \mathbf{s}_t & \mathbf{R}_t \end{pmatrix} \right]$$

where $\mathbf{s}_t = Cov[g(\eta_t), \boldsymbol{\theta}_t \mid D_{t-1}] = \mathbf{R}_t \mathbf{F}_t$.

The procedure adopted by West et. al. was to take $\mathbf{E}[\boldsymbol{\theta}_t \mid \eta_t\,, D_{t-1}]$ to be of the form

$$\mathbf{E}[\boldsymbol{\theta}_t \mid \eta_t, D_{t-1}] = \mathbf{d}_0 + \mathbf{d}_1 g(\eta_t) \equiv \mathbf{d}$$

where $\mathbf{d}_0$ and $\mathbf{d}_1$ should be chosen to minimise the sum of variances, that is

$$trace\,\{\mathbf{E}[A_t(\mathbf{d}) \mid D_{t-1}]\,\}$$

where the expectation is with respect to $\eta_t \mid D_{t-1}$, and

$$A_t(\mathbf{d}) = \mathbf{E}[\,(\theta_t - \mathbf{d})(\theta_t - \mathbf{d})^T \mid \eta_t\,, D_{t-1}]$$

If this minimum is achieved at $\hat{\mathbf{d}}$, a natural estimator of $Var[\boldsymbol{\theta}_t \mid \eta_t, D_{t-1}]$ is $A_t(\hat{\mathbf{d}})$. Performing this minimisation yields

$$\begin{aligned} \hat{\mathbf{d}} &= \left(\frac{\mathbf{a}_t - f_t \mathbf{s}_t}{q_t} \right) + \left(\frac{\mathbf{s}_t}{q_t} \right) g(\eta_t) \\ &= \mathbf{a}_t + \mathbf{s}_t (g(\eta_t) - f_t)/q_t \end{aligned}$$

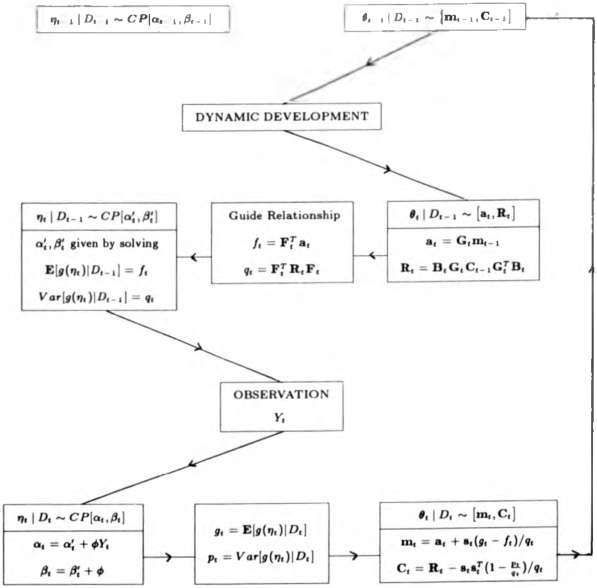
and $\qquad A_t(\hat{\mathbf{d}}) = \mathbf{R}_t - (\mathbf{s}_t \mathbf{s}_t^T)/q_t$

Substituting these into (2.3.14) gives

$$\mathbf{m}_t = \mathbf{a}_t + \mathbf{s}_t(g_t - f_t)/q_t \tag{2.3.15}$$
$$\mathbf{C}_t = \mathbf{R}_t - (\mathbf{s}_t\mathbf{s}_t^T)(1 - p_t q_t)/q_t$$

where $g_t = \mathbf{E}[g(\eta_t) \mid D_t]$ and $p_t = Var[g(\eta_t) \mid D_t]$ which can be computed from the full distribution of $\eta_t \mid D_t$ given in (2.3.13).

The full scheme of updating parameters is given in figure 2.3.16



$\eta_{t-1} \mid D_{t-1} \sim CP[\alpha_{t-1}, \beta_{t-1}]$
$\theta_{t-1} \mid D_{t-1} \sim [\mathbf{m}_{t-1}, \mathbf{C}_{t-1}]$
DYNAMIC DEVELOPMENT
$\eta_t \mid D_{t-1} \sim CP[\alpha_t', \beta_t']$
α_t', β_t' given by solving
$\mathbf{E}[g(\eta_t) \mid D_{t-1}] = f_t$
$Var[g(\eta_t) \mid D_{t-1}] = q_t$
Guide Relationship
$f_t = \mathbf{F}_t^T \mathbf{a}_t$
$q_t = \mathbf{F}_t^T \mathbf{R}_t \mathbf{F}_t$
$\theta_t \mid D_{t-1} \sim [\mathbf{a}_t, \mathbf{R}_t]$
$\mathbf{a}_t = \mathbf{G}_t \mathbf{m}_{t-1}$
$\mathbf{R}_t = \mathbf{B}_t \mathbf{G}_t \mathbf{C}_{t-1} \mathbf{G}_t^T \mathbf{B}_t$
OBSERVATION
Y_t
$\eta_t \mid D_t \sim CP[\alpha_t, \beta_t]$
$\alpha_t = \alpha_t' + \phi Y_t$
$\beta_t = \beta_t' + \phi$
$g_t = \mathbf{E}[g(\eta_t) \mid D_t]$
$p_t = Var[g(\eta_t) \mid D_t]$
$\theta_t \mid D_t \sim [\mathbf{m}_t, \mathbf{C}_t]$
$\mathbf{m}_t = \mathbf{a}_t + \mathbf{s}_t(g_t - f_t)/q_t$
$\mathbf{C}_t = \mathbf{R}_t - \mathbf{s}_t \mathbf{s}_t^T (1 - \frac{p_t}{q_t})/q_t$

FIGURE 2.3.16

2.4 The Beta–Binomial DGLM.

In this section we review a specific DGLM, the Beta–Binomial. The reason for this is to provide a background for sections 4.4 and 4.5 which are devoted to developing and applying a multivariate analogue to this model.

The model is specified as follows. Observations come from the likelihood

$$\begin{aligned} p(Y_t \mid \mu_t, n_t) &= \binom{n_t}{Y_t} \mu_t^{Y_t} (1-\mu_t)^{n_t - Y_t} \\ &= \binom{n_t}{Y_t} exp\left\{ Y_t \log\left(\frac{\mu_t}{1-\mu_t}\right) + n_t \log(1-\mu_t) \right\} \end{aligned} \tag{2.4.1}$$

which, by comparison with the exponential family likelihood (2.3.1), immediately gives the natural parameterisation as

$$\eta_t = \log\left(\frac{\mu_t}{1-\mu_t}\right)$$

If one now takes the guide function, $g(.)$, to be the identity, i.e. $g(x) = x$, then the model is specified by

$$\begin{aligned} \eta_t = \log\left(\frac{\mu_t}{1-\mu_t}\right) &= \mathbf{F}_t^T \boldsymbol{\theta}_t \\ \boldsymbol{\theta}_t &= \mathbf{G}_t \boldsymbol{\theta}_{t-1} + \mathbf{W}_t \end{aligned}$$

which can be described as a dynamic version of the standard logisitic–linear regression model.

The conjugate prior for η_t in the likelihood (2.4.1) is the Logistic–Beta distribution, $LBe(\alpha_t, \beta_t)$, with p.d.f.

$$p(\eta_t \mid \alpha_t, \beta_t) = \frac{\Gamma(\alpha_t + \beta_t)}{\Gamma(\alpha_t)\Gamma(\beta_t)} \frac{e^{\alpha_t \eta_t}}{(1 + e^{\eta_t})^{\alpha_t + \beta_t}}$$

and cumulants

$$\begin{aligned} \mathbf{E}[\eta_t] &= \gamma(\alpha_t) - \gamma(\beta_t) \\ Var(\eta_t) &= \dot{\gamma}(\alpha_t) + \dot{\gamma}(\beta_t) \end{aligned}$$

where γ is the digamma function and $\dot{\gamma}$ the trigamma function defined by

$$\begin{aligned} \gamma(\alpha) &= \frac{\mathrm{d}}{\mathrm{d}\alpha} \log \Gamma(\alpha) \\ \dot{\gamma}(\alpha) &= \frac{\mathrm{d}}{\mathrm{d}\alpha} \gamma(\alpha) \end{aligned}$$

Also one can compute

$$mode(\eta_t) = \log\left(\frac{\alpha_t}{\beta_t}\right)$$

and the curvature of $p(\eta_t \mid \alpha_t, \beta_t)$ at the mode as

$$\frac{\alpha_t \beta_t}{\alpha_t + \beta_t}$$

Refering to figure 2.3.16, assume we have set $\mathbf{m}_{t-1}, \mathbf{C}_{t-1}$ and hence obtained $\mathbf{a}_t, \mathbf{R}_t$ and, from the guide relationship, f_t and q_t. The next task is to obtain α_t' and β_t' by solving

$$f_t = \gamma(\alpha_t') - \gamma(\beta_t')$$
$$q_t = \dot{\gamma}(\alpha_t') + \dot{\gamma}(\beta_t')$$

this however requires numerical solution. So West et. al., pointing out that the guide relation really is just a guide, suggest, as an approximation, taking f_t as the mode of $\eta_t \mid D_{t-1}$ and q_t as the inverse of the curvature of $p(\eta_t \mid D_{t-1})$ at the mode. This gives the simple formula

$$\alpha_t' = \frac{1}{q_t}(1 + e^{f_t})$$
$$\beta_t' = \frac{1}{q_t}(1 + e^{-f_t})$$

Moving down figure 2.3.16 the updating from α_t', β_t' to α_t, β_t is given by

$$\alpha_t = \alpha_t' + Y_t$$
$$\beta_t = \beta_t' + n_t - Y_t$$

Note that this appears slightly inconsistent with figure 2.3.16, this is simply due to using the parameterisation α_t, β_t to make the Logistic–Beta be in its standard form. To be consistent with figure 2.3.16 we would have to use the parameterisation α_t, $(\alpha_t + \beta_t)/n_t$.

The values for g_t and p_t are given by

$$g_t = \gamma(\alpha_t) - \gamma(\beta_t)$$
$$p_t = \dot{\gamma}(\alpha_t) + \dot{\gamma}(\beta_t)$$

All other updating is as in figure 2.3.16.

An interesting special case occurs when $\mathbf{G}_t = \mathbf{I}$ and $\mathbf{F}_t$ is the same for two succesive time periods. In this case one way of modelling a dynamic development directly on η_t would be to use the power steady model, resulting in the updating

$$\alpha_t' = k_t \alpha_{t-1}$$

$$\beta_t' = k_t \beta_{t-1}$$

It is easily shown that, in this special case, the Beta–Binomial DGLM is equivalent to the power steady model with

$$\begin{aligned} k_t &= \frac{Var(\eta_{t-1} \mid D_{t-1})}{Var(\eta_t \mid D_{t-1})} \\ &= \frac{\mathbf{F}^T \mathbf{C}_{t-1} \mathbf{F}}{\mathbf{F}^T \mathbf{B}_t \mathbf{C}_{t-1} \mathbf{B}_t \mathbf{F}} \end{aligned}$$

where $\mathbf{F}$ is the common value of $\mathbf{F}_{t-1}$ and $\mathbf{F}_t$.

3. A REVIEW OF DECISION THEORETIC BIDDING MODELS

As noted in chapter 1 the two main models proposed are due to Friedman (1956) and Gates (1967). It should be emphasised that both these models were proposed primarily for bidding on one–off contracts, and not for sequential bidding.

The mathematical framework in which both these models exist is as follows. We represent company, or firm, 0, F_0, competing against companies $F_1, \ldots, F_n$. As described in chapter 1 our task is to compute $\hat{p}(m)$, an estimate of $p(m)$, where

$$p(m) = P(\text{we win the contract with markup } m)$$

The Friedman and Gates models both propose $\hat{p}(m)$ should be expressed as a function of $\hat{p}_1(m), \ldots, \hat{p}_n(m)$, where $\hat{p}_i(m)$ is an estimate of

$$p_i(m) = P(\text{we 'beat' } F_i \text{ with markup } m)$$

precisely what is meant by 'beats' is discussed in a moment. Explicitly, the Friedman model proposes,

$$\hat{p}(m) = \prod_{i=1}^{n} \hat{p}_i(m) \tag{3.1}$$

and the Gates model proposes,

$$\hat{p}(m) = \left[1 + \sum_{i=1}^{n} \frac{1 - \hat{p}_i(m)}{\hat{p}_i(m)}\right]^{-1} \tag{3.2}$$

The merits of these two models have been discussed extensively in the literature, resulting in a supposed controversy as to which is best to use. Before mentioning some of the main contributions to this controversy it is worth highlighting an unfortunate recurring trait of papers in this area. Most authors feel they have to answer the question "which of Gates' or Friedman's formulae is *right* ?". Surely this is the wrong question to be asking. Almost any formula can be deemed to be *right* if we make the appropriate assumptions. The questions should be, firstly - which assumptions vindicate the Friedman formula and which the Gates formula ? and, secondly - which of these assumptions are most likely to be appropriate in our given situation ? One of the aims of §4.1 is to isolate the underlying modelling assumptions and implications if the Gates formula is to be valid, so that the practitioner can determine whether or not it is appropriate to use it. This point has also been made by King & Mercer (1987,88) who attempt to isolate the assumptions underlying the Friedman and Gates models from a more practical viewpoint.

Attempts to answer the question of whether the Friedman or Gates formula is right, have been both experimental (i.e. based on performance in simulation) and theoretical. Theoretical attempts to 'prove' the Gates formula right have come from Rosenshine (1972), who also provides a good review of the controversy, and Dixie (1974). However Fuerst (1976) has highlighted errors in both these proofs. Fuerst goes on to make some interesting points about the Gates formula, which we mention in a moment.

Gates (1976) and Benjamin & Meador (1979) have performed simulations to try to justify formula (3.2). The Gates simulation was summarily dismissed by Fuerst (1977). Benjamin & Meadors simulation was very similar to Gates' and so many of Fuerst's criticisms also apply here. They also failed to take in to account the comments made by Fuerst (1976), on what might be a correct application of the Gates model. It is worth noting that in all of these simulations the Gates model performed significantly better than the Friedman model.

Before commenting further on (3.1) and (3.2) we define two sets of random variables. Let X_i be a random variable representing the bid made by company i, $i = 1 \ldots n$, on a given contract. Also, for a markup m, define the random variable $\theta_i(m)$ by

$$\theta_i(m) = p(F_i \text{ wins the contract} \mid \text{ we use markup } m)$$

We can thus model the problem by working with either the vector of bids $\mathbf{X} = (X_1, \ldots X_n)$ or the vector of probabilities $\boldsymbol{\theta}(m) = (\theta_0(m), \ldots, \theta_n(m))$, that is, in practice, start by putting a prior on either of $\mathbf{X}$ or $\boldsymbol{\theta}(m)$. In §4.1 we show that the Gates formula (3.2) follows directly from a Dirichlet prior on $\boldsymbol{\theta}(m)$.

The Friedman formula (3.1) clearly arises from putting independent priors on $X_1, \ldots, X_n$ and interpreting $p_i(m)$ as $p(cm < X_i)$, where c is our point estimate of the cost of the contract to us.

The above comments on the Friedman and Gates models lead us to make the following definitions. A '***Friedman type***' model is one which puts a prior directly on the random variable $\mathbf{X}$. Whilst a '***Gates type***' model is one which puts a prior directly on $\boldsymbol{\theta}(m)$.

Thus chapter 4 presents a sequential Gates type model, and chapter 5 presents a sequential Friedman type model.

Gates' own justification for his model was as follows. Assume that company F_i has s_i balls in an urn, and that the contract is awarded by drawing a ball at random from the urn. Then $p(m) = s_0/\sum_{i=0}^{n} s_i$, where s_0 is the number of balls we have in the urn. If the relationship $p_i(m) = s_0/(s_0 + s_i)$, is assumed, Gates' formula follows by substituting $s_i = s_0(1 - p_i(m))/p_i(m)$, in the formula for $p(m)$.

This justification inspired Fuerst (1976) to make the correct, and indeed obvious, deduction that if we know the exact distribution of $X_1, \ldots, X_n$ then $p_i(m)$ should be interpreted as

$$\begin{aligned} p_i(m) &= P(F_0 \text{ beats } F_i \text{ with markup } m \mid F_0 \text{ or } F_i \text{ win the contract}) \\ &= P(cm < X_i \mid F_0 \text{ or } F_i \text{ win the contract}) \end{aligned}$$

Since with this interpretation the Gates formula is a probabilistic identity. This interpretation of $p_i(m)$ is also obvious from the formalisation in §4.1. It is however at odds with the interpretation originally intended by Gates and used by Gates (1976) and Benjamin & Meador (1979) in their simulations. In these they simply took $p_i(m) = P(cm < X_i)$. For fixed m use of this interpretation of $p_i(m)$ is equivalent to assuming that the event of F_0 beating F_i is independent of the performance of F_0 and F_i relative to the other bidders. In chapter 6, theorem 6.1.4, it is proved that, provided $P(X_i = X_j) = 0$, there is no countably additive distribution across X_i, X_j such that for all x in a given range $P(x < X_i) = P(x < X_i \mid X_i < X_j)$. We can thus conclude that Gates' formula is never a probabilistic identity, for all m, with Gates' original interpretation of $p_i(m)$.

Decision theoretic models other than those of Friedman and Gates have come from, amongst others, Agnew (1972), Curtis & Maines (1974), Attanasi & Johnson (1975), Gunter & Swanson (1978), Carr (1982) and Knode & Swanson (1987). Agnew is one of the few to specifically tackle the sequential problem. He uses a novel non-parametric approach which starts with just a point estimate of the maximum of the profit function $(m-1)p(m)$. This estimate is then sequentially updated using techniques of stochastic approximation. The only information required for the updating is whether or not we win each contract.

Curtis & Maines and Carr claim to have broken free from the influence of the Friedman and Gates models by placing much more emphasis on the variability of each competitors cost estimates. Claiming, perhaps justifiably, that in many cases this variability is more important than variability in markup in accounting for variability in the final bid. However in both these papers the methods used to compute our probability of winning rely on the independence of competitors bids and in principle differ little from the Friedman approach.

The papers of Attanasi & Johnson, Gunter & Swanson and Knode & Swanson are representitive of a different, and important, approach to the problem. Their main thrust is that non-price factors such as a competitors resources are at least as important as profit in determining a competitors bid. They thus formulate a stochastic control model to maximise expected profit up to some finite horizon, given that when a contract is won this will affect a competitors ability to bid competitively at future time periods.

Non-price factors such as resources are discussed in more detail in Ward & Chapman (1988) and King & Mercer (1985).

An extensive bibliography containing over 500 references on all aspects of the bidding problem has been compiled by Stark & Rothkopf (1979). Most of the references however relate to very specific bidding environments such as defence or oil industry contracts, and tend to recommend very tailor made bidding procedures, relying little on general mathematical models.

Ultimately it is of course desirable to incorporate aspects of of these tailor made procedures, as well as non-price considerations, in to the more mathematical models. One of the features of the model in chapter 4 is that, although primarily a mathematical model, its framework does permit incorporation of other information. This is discussed in sections 4.3 and 4.5.

4. A SEQUENTIAL GATES TYPE MODEL

In this chapter we assume ourselves to be assisting an individual bidder in determining his bids on a sequence of contracts. The basic assumptions are those outlined in chapter 1, namely that $C \perp\!\!\!\perp p(m)$, and that the tenderer will award the contract to the bidder making the lowest markup.

The model outlined in sections 4.1 to 4.3 and generalised in sections 4.4 and 4.5 can work on the assumption that the only information our bidder receives after each contract is the value of the lowest markup, or bid, made by the other bidders. The identity of the winner when we do not win can be incorporated, but is not needed in updating our estimate of $p(m)$. Other information, such as the identity of the other bidders or the actual values of their bids, is not used in the model as stated. However the model could be extended to incorporate such information. Thus, as it stands, this model would be useful in an environment where we would not expect to receive much information in the aftermath of a contract being awarded.

4.1 A Formalisation of the Gates Model.

In the last chapter Gates' own justification for his model was described. In this section we construct distributional assumptions which lead to Gates' formula and confirm Fuerst's observations about the interpretation of the probabilities $p_i(m)$. These distributional assumptions are then shown to lead naturally to a sequential version of the Gates model. In the next section it is shown how this sequential model is easily extended to a simple power steady model of the form discussed in chapter 2. In §4.3 ways of incorporating some of problems faced by practicioners in to this model are discussed, methods for setting prior parameters are given, and finally there is an illustrative example.

Sections 4.4 and 4.5 show how the model can be further extended to a Dynamic Generalised Linear Model, allowing modelling and on line estimation of seasonal and regressor effects.

The first step in our formalisation of the Gates model is to define our random variables. For a given markup m let

$$\theta_i(m) = p(F_i \text{ wins the contract} \mid \text{we use markup } m) \qquad i = 0, \ldots, n$$

$\theta(m) = (\theta_0(m), \dots, \theta_n(m))$ is simply an unknown parameter vector, and so, as in any Bayesian analysis, we start by putting a prior distribution on it. Since somebody must win, $\sum_{i=0}^{n} \theta_i(m) = 1$. Thus consider the prior,

$$(\theta_0(m), \theta_1(m), \dots, \theta_n(m)) \sim \mathcal{D}(\alpha_0(m), \alpha_1(m), \dots, \alpha_n(m)) \tag{4.1.1}$$

where $\mathcal{D}((\alpha_0(m), \alpha_1(m), \dots, \alpha_n(m))$ represents the Dirichlet distribution with parameters $\alpha_0(m), \alpha_1(m), \dots, \alpha_n(m)$. Of course $p(m) = \theta_0(m)$, and the natural estimate of $\theta_0(m)$ is its expectation, thus take

$$\hat{p}(m) = \mathbf{E}[\theta_0(m)] = \frac{\alpha_0(m)}{\sum_{i=0}^{n} \alpha_i(m)} \tag{4.1.2}$$

Suppressing the dependancy on the index m for the time being, we now transform to new parameters.

Let $\psi_i = \theta_0/(\theta_0 + \theta_i)$, then it is easily shown that $(\psi_i, 1 - \psi_i) \sim \mathcal{D}(\alpha_0, \alpha_i)$, i.e a Beta distribution (see De Groot (1970)), so $\mathbf{E}[\psi_i] = \alpha_0/(\alpha_0 + \alpha_i) = \hat{\psi}_i$, say, where $\hat{\psi}_i$ is an estimate of ψ_i. Therefore, transforming to new parameters $\hat{\psi}_i$, $\alpha_i = \alpha_0(1 - \hat{\psi}_i)/\hat{\psi}_i$, substituting this in to (4.1.2) yields

$$\hat{p}(m) = \mathbf{E}[\theta_0] = \left[1 + \sum_{i=1}^{n} \frac{1 - \hat{\psi}_i}{\hat{\psi}_i}\right]^{-1} \tag{4.1.3}$$

Notice that $\hat{\psi}_i$ has the role of $\hat{p}_i$ in Gates' formula (3.2). To interpret $\hat{\psi}_i(m)$ is thus to interpret $\hat{p}_i(m)$ in the Gates formula.

Clearly since $\psi_i = \theta_0/(\theta_0 + \theta_i)$ we have, directly from the definition of the θ's,

$$\begin{aligned} \psi_i(m) &= \frac{p(F_0 \text{ wins contract})}{p(F_0 \text{ or } F_i \text{ wins the contract})} \\ &= p(F_0 \text{ wins the contract} \mid F_0 \text{ or } F_i \text{ wins the contract}) \end{aligned} \tag{4.1.4}$$

So as Fuerst(1976) points out, the Gates formula is valid provided $p_i(m)$ is interpreted as $\psi_i(m)$ above. We note this is not the usual interpretation of $p_i(m)$. To obtain these values in a practical environment, we must ask the question, "Given only us or company i can now win the contract, what is the probability that we win ? ".

We can thus conclude that Gates' formula is consistent with a Dirichlet distribution on $\theta(m)$.

The Dirchlet assumption (4.1.1) suggests a way in which the record of success of the companies $F_1, \dots, F_n$ in previous similar contracts can be incorporated to obtain improved estimates of $\mathbf{E}[\theta_0]$. To motivate this process, assume first that we always chose the same markup m. Initially choose $\alpha_0, \alpha_1, \ldots, \alpha_n$ so that they are consistent with our beliefs about $p_i(m)$. How this might be done in practice is discussed in §4.3. Now consider the award of the contract as a Bernoulli trial with $n+1$ types of success $0, 1, \ldots, n$. A type i success occurs with probability θ_i and is equivalent to company F_i winning the contract. Thus

$$p(r_0, r_1, \ldots, r_n \mid \boldsymbol{\theta}) = \prod_{i=0}^{n} \theta_i^{r_i} \tag{4.1.5}$$

where r_i is the number of type i successes, or equivalently,

$$r_i = \begin{cases} 1 & \text{if company } i \text{ has won} \\ 0 & \text{otherwise} \end{cases}$$

Clearly $p(\boldsymbol{\theta} \mid \mathbf{r}) \propto p(\mathbf{r} \mid \boldsymbol{\theta}) p(\boldsymbol{\theta})$ thus from (4.1.1) and (4.1.5)

$$(\boldsymbol{\theta}(m) \mid \mathbf{r}) \sim \mathcal{D}(\alpha_0^*(m), \alpha_1^*(m), \ldots, \alpha_n^*(m)) \tag{4.1.6}$$

where $\alpha_i^*(m) = \alpha_i(m) + r_i$, in particular,

$$\hat{p}(m \mid \mathbf{r}) = \frac{\alpha_0^*(m)}{\sum_{i=0}^{n} \alpha_i^*(m)}$$

and,

$$\hat{p}_i(m \mid \mathbf{r}) = \frac{\alpha_0^*(m)}{\alpha_0^*(m) + \alpha_i^*(m)}$$

So the relevant probabilities can be updated directly once we learn who won the last contract.

We have now established how, for fixed m, we can update the probabilities $\hat{p}(m)$ and $\hat{p}_i(m)$ when we know the identity of the winner, F_k say, of the last contract. In order that we can update these probabilities for all m, suppose we are told that after all the bids had been submitted, the lowest markup made by our competitors was m^*, and that the identity of the competitor using this markup was F_k. We now know that for all fixed markups m less than m^* we would have won the contract, and thus would have the following updating, $\alpha_0^*(m) = \alpha_0(m) + 1$ and $\alpha_i^*(m) = \alpha_i(m)$ for all $i \neq 0$. And for all fixed markups m greater than m^* we would have, $\alpha_k^*(m) = \alpha_k(m) + 1$ and $\alpha_i^*(m) = \alpha_i(m)$ for all $i \neq k$. Suppressing the index m on $\alpha_0, \ldots, \alpha_n$ we therefore have the following updating for our probabilities:

$$\hat{p}(m \mid m^*) = \begin{cases} [1 + \alpha_0] \,/\, [1 + \sum_{i=0}^{n} \alpha_i] & m < m^* \\ \alpha_0 \,/\, [1 + \sum_{i=0}^{n} \alpha_i] & m > m^* \end{cases} \tag{4.1.7}$$

$$\hat{p}_i(m \mid m^*, r_i) = \begin{cases} [\alpha_0 + 1] / [\alpha_0 + \alpha_i + 1] & m < m^* \\ \alpha_0 / [\alpha_0 + \alpha_i + r_i] & m > m^* \end{cases} \tag{4.1.8}$$

and the full distribution of $\theta(m)$ can be updated from

$$\left.\begin{array}{c} \alpha_0(m) = \alpha_0(m) + 1 \\ \alpha_i(m) = \alpha_i(m) \end{array}\right\} \qquad m < m^*$$

$$\left.\begin{array}{c} \alpha_0(m) = \alpha_0(m) \\ \alpha_i(m) = \alpha_i(m) + r_i \end{array}\right\} \qquad m > m^*$$

A practical problem arises if we are only told the value of the winning markup, rather than m^*. Although we can use the updating in (4.1.7) and (4.1.8) when we do not win the contract, if we do win, with a bid $m^{(1)}$ say, we know only that $m^* \geq m^{(1)}$. Clearly updating of $\hat{p}(m)$ is unaffected for $m \leq m^{(1)}$, otherwise, we may have to opt for the overly conservative updating obtained by setting $m^* = m^{(1)}$.

It has been shown that if we know the value of our competitors' lowest markup after each contract, it is only neccesary to estimate $\hat{p}_i(m)$ at the first time period. From then on $\hat{p}(m)$ and $\hat{p}_i(m)$ can be updated directly, given only the minimal information of our competitors lowest markup and the identity of the company making this markup – information which is often easily accessible.

To obtain an understanding of how this model works in the long run, it is helpful to consider the behaviour of $\hat{p}(m)$ and $\hat{p}_i(m)$ as the number, N, of our competitors best markups we have observed gets large. Let these markups be denoted by $m_1^*, \ldots, m_N^*$, and assume that none are equal, so, without loss of generality, we can assume they are ordered i.e. $m_1^* < m_2^* <, \ldots, < m_N^*$. Repeated application of (4.1.7) yields

$$\hat{p}(m \mid m_1^*, \ldots, m_N^*) = \frac{\alpha_0 + N - r}{\sum_{i=0}^{n} \alpha_i + N} \qquad m \in (m_r^*, m_{r+1}^*)$$

for $r = 0, \ldots, N$, where we define $m_0^* = -\infty$ and $m_{N+1}^* = +\infty$. As N gets large we have,

$$\hat{p}(m \mid m_1^*, \ldots, m_N^*) \simeq \frac{N - r}{N} \qquad m \in (m_r^*, m_{r+1}^*)$$

i.e. the probability that we win with a markup in the range (m_r^*, m_{r+1}^*), tends to the proportion of times we would have won, if we had used a markup in this range at all time periods. This is a pleasingly sensible result, and fits in well with Gates' original interpretation of his model. We also note that $var(\theta_0) \simeq r(N - r)/N^2(N + 1)$ as N gets large, thus $var(\theta_0) \to 0$ as $N \to \infty$. So we can state the stronger result that the random variable θ_0 tends to $(N-r)/N$, in probability, as $N \to \infty$. We also find,

$$\tilde{p}_i(m \mid m_1^*, \ldots, m_N^*) \simeq \frac{N-r}{N+R_i-r} \qquad m \in (m_r^*, m_{r+1}^*)$$

where R_i is the number of contracts that company i would have won, if we had used a markup in the range (m_r^*, m_{r+1}^*) at every time period. Thus $\sum_{i=1}^{n} R_i = r$, since $N-r$ contracts would have been won by us. Exactly analogous results pertain if some of the observed markups are identical.

One suprising feature of the updating formula (4.1.7) is that $\tilde{p}(m)$ can be updated without knowing the identity of our best competitor on each contract so far. Thus, one implication of the Dirichlet model is that given no other information about our competitors, the sequence of their lowest bids on each contract is sufficient for the prediction of $p(m)$. However, if as is usual, we expect to gain information about who else has been invited to quote for a particular contract, then the identities of those competitors making winning bids is crucial to our updated probability $\tilde{p}(m)$. For example if we have been told that we are only competing against companies $F_1, \ldots, F_k$, $k < n$, then we have implicitly been told that $F_{k+1}, \ldots, F_n$ cannot win. Thus when working with the distribution of $\theta_0(m), \ldots, \theta_k(m)$ we must condition on the event $Y = \{\theta_{k+1}(m) = 0, \ldots, \theta_n(m) = 0\}$. From De Groot (1970) we have

$$(\theta_0(m), \ldots, \theta_k(m) \mid Y) \sim \mathcal{D}(\alpha_0^*(m), \ldots, \alpha_k^*(m))$$

where currently $\theta(m) \sim \mathcal{D}(\alpha_0^*(m), \ldots, \alpha_n^*(m))$. So in particular, when we have observed a number of markups, the probability that F_0 wins is now given by

$$\tilde{p}(m \mid Y) = \frac{\alpha_0 + S_0}{\sum_{i=1}^{m}(\alpha_i + S_i)}$$

where S_i is the total number of contracts won so far by F_i. This illustrates how different types of information can be utilised to improve $\tilde{p}(m)$, as emphasised by Ward & Chapman (1988).

4.2 A Steady Model Version.

An important practical problem in estimating probabilities is that the policies of competing companies are likely to drift with time, see Ward & Chapman (1988). Possible reasons for this are changes in management personnel or management policy, or a company diversifying its interests. Because of this, a competitor's behaviour on more recent contracts is often more relevant for making predictions about its future behaviour than what it did in the distant past. In the last model we have implicitly assumed that probabilities $\theta_i(m)$ are static, i.e. they do not change with time. This leads, for example, to the unrealistic belief that given a large enough N we can be almost certain about our chances of winning with a given markup m. We can model the probability $\boldsymbol{\theta}(t)$ steadily changing over time by using the Simple Power Steady Model of chapter 2. i.e. take

$$f_{t+1}(\boldsymbol{\theta}(t+1) \mid \mathbf{r}(t)) \propto [f_t(\boldsymbol{\theta}(t) \mid \mathbf{r}(t))]^k \tag{4.2.1}$$

where $k \in (0,1]$, and $f_t(.)$ is the p.d.f of $\boldsymbol{\theta}(t) \mid .$ at time t. Note, for convenience we are now indexing $\boldsymbol{\theta}$ by time, t, although it is still of course a function of m. This development means that individual decisions associated with $\boldsymbol{\theta}$ are unchanged. However any uncertainty associated with these decisions will increase at each time period. Of course if $k = 1$ we have the static case considered previously. For general k it is easily checked that this development implies

$$(\boldsymbol{\theta}(t+1) \mid \mathbf{r}(t)) \sim \mathcal{D}(\, k\boldsymbol{\alpha}(t) + 1 - k\,) \tag{4.2.2}$$

and

$$(\,\boldsymbol{\theta}(t+1) \mid \mathbf{r}(t+1)\,) \sim \mathcal{D}(\, k\boldsymbol{\alpha}(t) + \mathbf{r}(t+1) + 1 - k\,) \tag{4.2.3}$$

the analogues of (4.1.7) and (4.1.8) are,

$$\hat{p}(m \mid m^*) = \begin{cases} [k\alpha_0 + 2 - k] \,/\, [1 + (n+1)(1-k) + k\sum_{i=0}^{n} \alpha_i] & m < m^* \\ [k\alpha_0 + 1 - k] \,/\, [1 + (n+1)(1-k) + k\sum_{i=0}^{n} \alpha_i] & m > m^* \end{cases}$$

and

$$\hat{p}_i(m \mid m^*, r_i^*) = \begin{cases} [k\alpha_0 + 2 - k] \,/\, [3 - 2k + k(\alpha_0 + \alpha_i)] & m < m^* \\ [k\alpha_0 + 1 - k] \,/\, [2(1-k) + r_i + k(\alpha_0 + \alpha_i)] & m > m^* \end{cases}$$

To obtain a better understanding of how this model operates we now look at its behaviour in the long run, when the discount is constant between contracts. For a given m, after T contracts, the updating in (4.2.3) gives

$$\alpha_i(T) = k^T \alpha_i(0) + 1 - k^T + \sum_{t=0}^{T} k^t r_i(T-t)$$

where

$$r_i(t) = \begin{cases} 1 & \text{if company } i \text{ has won the contract at time } t \\ 0 & \text{otherwise} \end{cases}$$

Therefore as $T \to \infty$

$$\alpha_i(T) \to 1 + \sum_{t=0}^{\infty} k^t r_i(T-t) \qquad (4.2.4)$$

and thus

$$\bar{p}(m,T) = \mathbf{E}[\theta_0(T)] = \frac{\alpha_0(T)}{\sum_{i=0}^{n} \alpha_i(T)} \to \frac{1 + \sum_{t=0}^{\infty} k^t r_0(T-t)}{1 + n + (1-k)^{-1}}$$

since $\sum_{i=0}^{n} r_i(t) = 1$ for all t. Also, the mode of $\theta_0(T)$, $M[\theta_0(T)]$, say, is given by

$$M[\theta_0(T)] = \frac{\alpha_0(T) - 1}{\sum_{i=0}^{n} \alpha_i(T) - n - 1} \to (1-k)\sum_{t=0}^{\infty} k^t r_0(T-t)$$

So if we let $M_0 = (1-k)\sum_{t=0}^{\infty} k^t r_0(T-t)$, then M_0 can be interpreted as a discounted version of the true proportion of times we would have won contracts, if we had used this markup m at all time periods, recalling that $\theta_0(T) = \theta_0(T,m)$. Now we can write

$$\lim_{T\to\infty} \bar{p}(m,T) = \gamma(\frac{1}{n+1}) + (1-\gamma)M_0$$

where

$$\gamma = \frac{(1-k)(n+1)}{1 + (1-k)(n+1)} \in [0,1)$$

thus $\bar{p}(m,T)$ tends to a weighted average of the discounted true proportion and a state of complete uncertainity, represented by our probability of winning being $1/(n+1)$.

We also note, from (4.2.4), that

$$var[\theta_0(T)] \to \frac{[1 - k + M_0][1 + n(1-k) - M_0]}{[1 + (n+1)(1-k)]^2[1 + (n+2)(1-k)]}$$

So, unlike the static case, $var[\theta_0(T)] \not\to 0$ as $T \to \infty$. Hence we can make no statements about the convergence of the random variable θ_0. We have thus avoided the somewhat unrealistic consequence of the static Dirichlet model of becoming completely certain about our chances of winning with a markup m as the number of past contracts $T \to \infty$.

An alternative form of the steady model is to apply the evolution in (4.2.1) to a transformation of $\theta(t)$. In particular we might think of the multivariate logistic transform given by

$$\eta_i(t) = \log\left(\frac{\theta_i(t)}{\theta_0(t)}\right) \qquad i = 1, \ldots, n$$

(this is used more extensively in the Dynamic Generalised Linear Model of sections 4.4 and 4.5). Applying this steady evolution to $\eta(t)$ gives

$$(\theta(t+1) \mid \mathbf{r}(t)) \sim \mathcal{D}(\, k\alpha(t)\,)$$

and

$$(\theta(t+1) \mid \mathbf{r}(t+1)) \sim \mathcal{D}(\, k\alpha(t) + \mathbf{r}(t+1)\,)$$

the analogues of (4.1.7) and (4.1.8) are,

$$\bar{p}(m \mid m^*) = \begin{cases} [k\alpha_0 + 1] \,/\, [1 + k\sum_{i=0}^{n} \alpha_i] & m < m^* \\ [k\alpha_0] \,/\, [1 + k\sum_{i=0}^{n} \alpha_i] & m > m^* \end{cases}$$

and

$$\bar{p}_i(m \mid m^*, r_i^*) = \begin{cases} [k\alpha_0 + 1] \,/\, [1 + k(\alpha_0 + \alpha_i)] & m < m^* \\ [k\alpha_0] \,/\, [1 + k(\alpha_0 + \alpha_i)] & m > m^* \end{cases}$$

Once again looking at the asymptotic behaviour, we have after T contracts

$$\alpha_i(T) = k^T \alpha_i(0) + \sum_{t=0}^{T} k^t r_i(T-t)$$

therefore as $T \to \infty$

$$\alpha_i(T) \to \sum_{t=0}^{\infty} k^t r_i(T-t) \tag{4.2.4}$$

and thus

$$\bar{p}(m,T) \to \frac{\sum_{t=0}^{\infty} k^t r_0(T-t)}{(1-k)^{-1}} = M_0$$

So in this case $\bar{p}(t,m)$ tends to the discounted true proportion. Once again, of course, $var[\theta_0(T)] \not\to 0$ as $T \to \infty$ so we can make no comments about the convergence of the random variable θ_0. This alternative evolution illustrates nicely the flexibility of the steady model in allowing us to apply steady evolutions to transformations of random variables. The transformation of θ to η is particularly appropriate as many have argued that it is easier and more natural to work in terms of these log odds rather than probabilities, see for example Spiegelhalter & Knill–Jones (1986).

In the above two cases we have assumed that the drift in information, modelled by the parameter k, is the same between contracts. Although this may be a reasonable assumption to make if contracts appear at regular, or approximately regular, intervals of time, it is probably more realistic to relate k to the waiting time between contracts, with more drift of information occuring over longer time periods. The simplest way to do this is to let

$$1 - k = (1-\lambda)\frac{\tau}{\tau_0} \tag{4.2.5}$$

Where τ is the time interval between the tendering of the last contract and the tendering of this one, and τ_0 is some *base* time period, for example, the average time between contracts. Thus k is now a function of τ, and λ is a constant we have to set. Note that under this scheme, if two contracts are put out simultaneously ($\tau = 0$) no discounting of information takes place.

Finally, consider setting the parameter λ. From (4.2.5) we have that λ is the value of k representing the amount of drift in θ that occurs in time τ_0. In the next section we explain how the parameter $d = \sum_{i=1}^{n} \alpha_i - n$ can be viewed as the number of data points we think our current setting of α is worth. If after time τ_0 we have observed no data, (4.2.1) tells us that d will be updated to λd. So we can set λ by answering the question 'after a further time period τ_0, what proportion of its present value will our current setting of α be worth ?'. This proportion will be λ, and will usually be in the range $(.8, 1)$.

4.3 Using The Model In Practice.

To use the model described in the previous two sections we must first set the prior parameters $\alpha_0(m), \ldots, \alpha_n(m)$. At first sight it seems we must specify $n+1$ functions of m, for example by putting a joint distribution across $X_1, \ldots, X_n$ and setting $\alpha_i(m)$ from the relationship $\alpha_i(m) = \alpha_0(m)(1-\bar{\psi}_i(m))/\bar{\psi}_i(m)$, $\alpha_0(m)$ being arbitarily specified. However we can avoid this by conditioning on the event we do *not* win, as follows. Let ϕ_i denote the probability that company i wins the contract given that we do not. Then it is easily shown that

$$\phi_i = \frac{\theta_i}{\sum_{j=1}^{n} \theta_j}$$

A plausible modelling assumption is that the distribution of $\phi_1, \ldots, \phi_n$ should not depend on our chosen markup m. From De Groot (1970) we have,

$$(\phi_1, \ldots, \phi_n) \sim \mathcal{D}(\alpha_1, \ldots, \alpha_n)$$

with $\alpha_1, \ldots, \alpha_n$ as in (4.1.1). So we conclude that the parameters $\alpha_1, \ldots, \alpha_n$ will not need to depend on m either. We also have

$$\mathbf{E}[\phi_i] = \bar{\phi}_i = \frac{\alpha_i}{\sum_{j=1}^{n} \alpha_j}$$

If we let $\beta = \sum_{i=1}^{n} \alpha_i$ then $\alpha_i = \beta\bar{\phi}_i$. So for a fixed β we can determine values for α_i, by estimating $\bar{\phi}_i$. Clearly $\bar{\phi}_i$ represents the proportion of contracts we would expect company i to win when we do not win. If we are completely uninformed about our competitors we could set $\bar{\phi}_i = 1/n$ for all i. In most cases though, we will presumably have some information about the past success rates of our competitors. This information could be used to fix our estimate of $\bar{\phi}_i$

as the proportion of contracts F_i would have won if F_0 were not in contention. The parameter β reflects our confidence in our prior settings of $\bar{\phi}_1, \ldots, \bar{\phi}_n$. Larger values of β correspond to greater confidence. To calibrate β, consider the case $\beta = n$ and $\bar{\phi}_i = 1/n$, so $\alpha_i = 1$ for $i = 1, \ldots, n$. This represents a state of complete ignorance about the ϕ_i's, since the Dirichlet distribution degenerates to a uniform distribution. As β increases the α_i's increase and we move away from the uniform. This suggests setting $\beta - n = d$ say, $d \geq 0$, where larger values of d correspond to more confidence about our prior setting of the ϕ_i's. In fact we can be more precise and set d by letting it represent the number of data points we think our prior beliefs are worth.

Consider now the remaining problem of assigning a prior value to $\alpha_0(m)$. We could simply use

$$\bar{\psi}_1(m) = \bar{p}_1(m) = \frac{\alpha_0(m)}{\alpha_1 + \alpha_0(m)}$$

However, direct assesment of $\bar{p}_1(m)$ is often difficult, and even if we put a prior distribution on $X_1, \ldots, X_n$ computation of $\bar{\psi}_1(m)$ is tedious. Since our prior value for $\bar{p}(m)$ is $\alpha_0(m)/[\beta + \alpha_0(m)]$, we can write $\alpha_0(m) = \beta \bar{p}(m)/[1 - \bar{p}(m)]$. So why not assign prior beliefs directly to $\bar{p}(m)$? A simple form would be

$$\bar{p}(m) = \begin{cases} 1 & m \leq L \\ \frac{L}{U-L}\left(\frac{U}{m} - 1\right) & m \in (L, U) \\ 0 & m \geq U \end{cases} \tag{4.3.1}$$

Here, we are certain to win with a markup of L or less, and certain to lose with a markup of U or more. In between, the probability of us winning with a markup m behaves as $1/m$. Of course if we have any information on our competitors it should be used to produce a more appropriate form for $\bar{p}(m)$.

At first glance it may appear we have come full circle, since we are now proposing to fix prior parameters by producing an estimate of $\bar{p}(m)$. However, provided we have not chosen d too large, after observing the first few data points our estimates of the probabilities will be insensitive to these initial settings. This is especially true with the steady models, which automatically discount prior information. What *is* important is to have a simple method for assigning prior parameters that allows us to incorporate intuitive beliefs, rather than requiring precise mathematical specifications.

Summarising, we have shown that it is only neccesary to specify prior information once, at the beginning. From then on our parameters are updated directly. It is required that prior values are given for the parameters $\hat{\phi}_1, \ldots, \hat{\phi}_n, d$ and $\bar{p}(m)$, all of these can be given practical interpretations and so are easy to set. Uninformative prior settings for these parameters have also been given, for use when no strong information is available about the behaviour of competing firms. Of course just because our model updates parameters automatically once they have been set apriori, this does *not* mean that we cannot intervene to adjust them in the light of relevant external information. This information could be, for example, that we discover one of our competitors is in desperate need of work. We discuss this further later in the section.

We conclude our discussion of prior parameters by looking at one implication of the Dirichlet assumption (4.1.1). Suppose we have opted to have a prior directly on $X_1, \ldots, X_n$ and thus maintain seperate values for $\alpha_1, \ldots, \alpha_n$. If at some stage we gain information about company F_j, then our estimate of $\hat{\psi}_i(m)$ $(i \neq j)$ will be unchanged, since $\hat{\psi}_i = \alpha_0/(\alpha_0 + \alpha_i)$. Suppose we believe that if company F_i has a reasonable chance of winning then so will company F_j. In the event of company F_i winning we would thus want both $\hat{\psi}_i$ and $\hat{\psi}_j$ to be decreased. In the current setup, only $\hat{\psi}_i$ would be decreased. One simple way of overcoming this discrepancy is to group together these *similar* companies, and put a Dirichlet model over the groups. For example if we have five competitors $F_1, \ldots, F_5$ we may group them as follows $\{F_1, F_2\}, \{F_3, F_4, F_5\}$. Thus (4.1.1) becomes $(\theta_0, \theta_1, \theta_2) \sim D(\alpha_0, \alpha_1, \alpha_2)$. Where $\theta_1 = p(F_1 \text{ or } F_2 \text{ wins})$ and $\theta_2 = p(F_3 \text{ or } F_4 \text{ or } F_5 \text{ wins})$. We then interpret $\psi_1(m)$ as,

$$\psi_1(m) = p(F_0 \text{ beats } F_1 \text{ and } F_2 \mid \text{ only } F_0, F_1 \text{ or } F_2 \text{ can win})$$

$\psi_2(m)$ is interpreted similarly. Our prior settings for α_0, α_1 and α_2 can be determined as before.

Another reason why we may wish to group companies together, is that the person putting the contract out to tender will often have grouped the companies in his own mind. Given he has done this, there are then practical and theoretical reasons to suggest that he should, as far as possible, ask just one company from each group to tender. Assuming our groupings will be similar to his, we can take advantage of this by grouping companies ourselves, and thus have far more efficient updating.

Consider now the problem of computing our optimal markup at each time period. A natural way of doing this is to maximise our expected percentage profit over m for the next contract. This profit function is always of the form

$$prof(m;r,N) = (m-1)\frac{k^N\alpha_0(m)+c_1(r,N)}{k^N\alpha_0(m)+c_2(N)} \qquad m\in(m_r^*,m_{r+1}^*)$$

where $c_1(r,N)$ and $c_2(r)$ depend on the form of the steady model we are using, and the order in which the markups $m_1^*,\ldots,m_N^*$ have been recorded. For example under the steady evolution applied directly to the Dirichlet distribution we have

$$c_1(r,N) = 1-k^N+\sum_{t=0}^{N}k^t r_0(N-t)$$

$$c_2(N) = k^N\beta+(N+1)(1-k^N)+\sum_{t=0}^{N}k^t$$

$prof$ is a piecewise function of m so we need to maximise it over each piecewise segment, i.e. for each r. Clearly for each interval (m_r^*,m_{r+1}^*), the maximum of $prof(m;r,N)$ will, more often than not, occur at one of the endpoints. The following result takes advantage of this to simplify our maximisation problem.

LEMMA 4.3.2.

Assume that for $m\geq 1$,

(1) *$\bar{p}(m)$ is apriori continuous and decreasing*

(2) *there exists r_1 such that $\frac{d}{dm}prof(m;r_1,N)\geq 0$ for all $m\in[1,m_{r_1+1}^*]$*

Then the maximum of $prof(m;r,N)$ in (m_r^*,m_{r+1}^*) is achieved at m_{r+1}^* for all $r\leq r_1$

PROOF:

Assumption (2) tells us that $\frac{d}{dm}prof(m;r_1,N)\geq 0$ for all $r\leq r_1$ with $m\in(m_r^*,m_{r+1}^*)$, since if $r\leq r_1$ the range (m_r^*,m_{r+1}^*) is in the range $[1,m_{r_1+1}^*]$. This is equivalent to

$$h(r_1;N,m)\equiv(m-1)(c_2(N)-c_1(r,N))\frac{d\alpha_0}{dm}+(\alpha_0+c_1(r,N))(\alpha_0+c_2(N))\geq 0$$
$$m\in(m_r^*,m_{r+1}^*)$$

clearly $h(r;N,m)$ is a decreasing function of r, since $\alpha_0\geq 0$, $\frac{d\alpha_0}{dm}\leq 0$ from assumption (1), and $c_1(r,N)$ is an increasing function of r. Therefore $h(r;N,m)\geq h(r_1;N,m)\geq 0 \quad \forall r\leq r_1$ and $\forall m\in(m_r^*,m_{r+1}^*)$, hence $prof(m;r,N)$ is an increasing function in (m_r^*,m_{r+1}^*) for all $r\leq r_1$. So the maximum of $prof(m;r,N)$ in (m_r^*,m_{r+1}^*) is achieved at m_{r+1}^* for all $r\leq r_1$. □

In practice the lemma suggests the following procedure. Maximise the profit function by solving $\frac{d}{dm}prof = 0$, for $r = N, N-1, \ldots$ until we find a value of r, r_1, for which $\frac{d}{dm}prof(m; r, N) \geq 0$ for all $m \in [1, m^*_{r+1}]$. From then on consider only the the profit function at the upper endpoints.

Before giving an example we discuss how some of the practical considerations outlined in §1.1 can be incorporated into the model.

(i) *More realistic utilities and Non-price considerations*. In the computation of the optimal markup above the criteria used was simply to maximise the expected profit. In practice other 'non-price' considerations may need to be looked at. Properties of a particular contract may make us want to change our bidding behaviour, for example we may want to bid more competitively for contracts we deem to be of a high prestige, or for contracts with a very high cost. These effects are best handled as regressors in the model of §4.4 and §4.5. However, considerations such as the amount of work we are currently engaged on can be handled by altering our utility to reflect the fact that we may want to bid less competitively when we are already engaged on a number of other jobs. For example we might take a utility of the form

$$U(m) = u(m, J)prof(m, r, N)$$

where J is the number of jobs we are currently engaged on and $u(m, J)$, which lies in $(0, 1]$ say, satisfies the following

(1) $u(m, J)$ is an increasing function of m

(2) $u(m, J)$ is a decreasing function of J

(3) $\frac{\partial u(m,J)}{\partial m}$ is a decreasing function of J

A utility of this form will make us use larger markups as J increases.

This idea can be extended to a stochastic control model if we consider how the event of us winning the present contract affects our bidding on future contracts. Our task now is to maximise our expected utility up to some finite time horizon. Note that this requires specification of a distribution on the duration of a job.

(ii) *Seasonality*. Like regressors such as prestige and the cost of the contract, seasonality can be handled in a sophisticated manner by using the models of §4.4 and §4.5. However, unlike these regressors, it is simple to build a seasonal effect in to the model as it stands.

One of the reasons for this is the systematic nature of the seasonality we expect to see in a bidding environment. This systematic behaviour is likely to be so because a firms bidding practices will clearly be influenced by first harmonic effects such as number of daylight hours, and temperature.

The method for modelling this first harmonic effect is based on the idea that a given markup in one month is equivalent, in terms of our probability of winning, to a different markup in another month. We can model this by having a separate probability vector, $\theta^t(m)$, for each month t, but relating them via the equation

$$\theta^t(m) = \theta\left(m + b\sin(\frac{2\pi(t-a)}{12})\right)$$

where a and b are fixed parameters. The main disadvantage of this model compared with the model of §4.5 is that it does not provide any formal framework for on line estimation of parameters.

(iii)*Intervention*. It should be stressed that the output from a simplistic model, such as that described in the previous two sections, should only be used as a guide for making decisions, and intervention by the operator should be encouraged whenever subjective information is received. In practice subjective information can easily be incorporated by altering the current settings of $\alpha_0, \ldots, \alpha_n$. For example we can easily reflect a general change in the whole bidding environment by decreasing β, to reduce our confidence in the current α values, and resetting $\alpha_1, \ldots, \alpha_n$, to leave α_i/β unchanged. A more detailed account of situations where intervention may be called for, and the form this intervention should take is given in §4.5.

Example.

This example is based on real data issued by Stratford on Avon district council. The contracts were for external painting of buildings during the calendar year 1987. We can assume each contract is bid for by four companies, excluding ourselves.

The analysis will proceed along the following lines,

(1) set prior parameters,

(2) estimate the cost of the forthcoming contract, c,

(3) compute the optimal markup, $\overline{m}$,

(4) observe the winning markup, and return to step (2).

For step (1) we shall use the uninformative priors mentioned in the previous section. So take $\phi_i = 1/4$ for $i = 1, \ldots, 4, d = 0$ and $p(m)$ as in (4.3.1) with $L = 1.1$ and $U = 1.3$, say. Since all the contracts occur in such a short space of time it is uneccessary to discount information, so we shall use the static model of §4.1.

Table 4.3.3 summarises the observed markups and our chosen markups for each of 31 contracts.

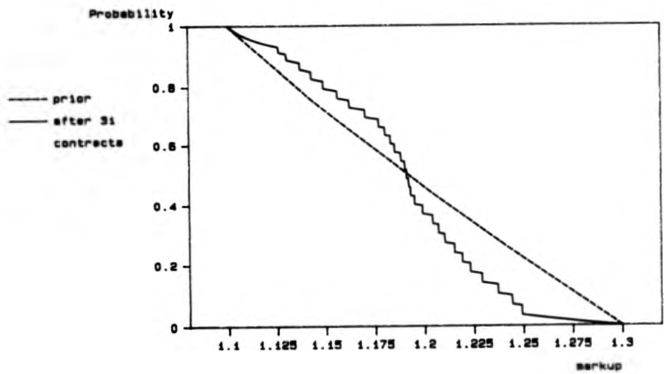
Figure 4.3.4 shows the probability we win with markup m both apriori and after the 31 contracts. The corresponding expected percentage profit functions are shown in figure 4.3.5.

We see that, at least initially, strict adherence to the model would result in us winning a lot of contracts. This suggests that a utility that discourages us from having too many jobs on the go at once may be appropriate. Note that on the 31^{st} contract $r_i = 26$ illustrating the computational saving of using lemma 4.3.2 when the number of observed markups gets large.

N	cost (£)	chosen markup	winning markup	m^*
0	18,000	1.140	1.126	1.126
1	11,000	1.126	1.126	1.192
2	11,500	1.126	1.126	1.191
3	6,000	1.126	1.126	1.244
4	7,500	1.191	1.191	1.210
5	9,250	1.191	1.191	1.193
6	8,000	1.191	1.191	1.229
7	12,000	1.191	1.191	1.199
8	11,750	1.191	1.191	1.223
9	15,500	1.191	1.149	1.149
10	10,000	1.191	1.162	1.162
11	8,000	1.191	1.191	1.207
12	14,750	1.191	1.143	1.143
13	14,000	1.143	1.135	1.135
14	12,000	1.126	1.126	1.170
15	11,750	1.126	1.126	1.188
16	13,500	1.135	1.135	1.156
17	11,000	1.135	1.135	1.180
18	10,250	1.135	1.135	1.204
19	9,000	1.135	1.135	1.185
20	9,500	1.143	1.143	1.195
21	7,500	1.143	1.143	1.249
22	11,250	1.143	1.143	1.190
23	9,500	1.143	1.143	1.219
24	8,750	1.143	1.143	1.237
25	10,750	1.143	1.143	1.183
26	10,000	1.143	1.143	1.215
27	9,750	1.180	1.177	1.177
28	14,500	1.156	1.130	1.130
29	13,750	1.156	1.156	1.183
30	12,000	1.170	1.170	1.174
31	9,000	1.174		

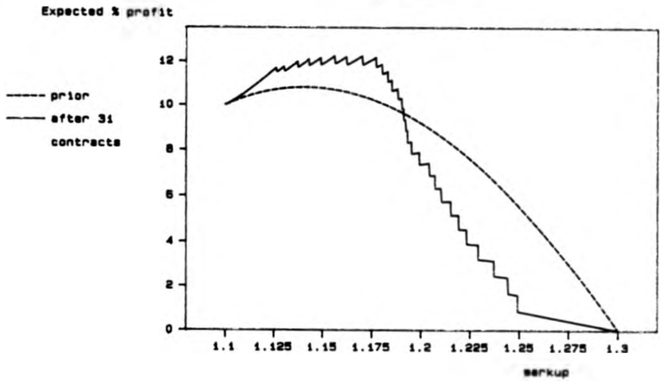
TABLE 4.3.3

FIGURE 4.3.4



Probability
1
0.8
0.6
0.4
0.2
0
1.1
1.125
1.15
1.175
1.2
1.225
1.25
1.275
1.3
markup
prior
after 31
contracts

FIGURE 4.3.5


Expected % profit
prior
after 31
contracts
12
10
8
6
4
2
0
1.1
1.125
1.15
1.175
1.2
1.225
1.25
1.275
1.3
markup

4.4 The Dirichlet–Multinomial DGLM.

In chapter 2 the DGLM of West et. al (1985) was reviewed, and the special case of the Beta–Binomial model was described in detail. In this section we concentrate on developing a multivariate analogue to the Beta–Binomial model, namely a Dirichlet–Multinomial model. Details of applying this to our sequential Gates type model are then discussed in the next section.

The basic setup is as follows. At time t let $n+1$ events have probabilities $\mu_0(t), \ldots, \mu_n(t)$, $\sum_{i=0}^{n} \mu_i(t) = 1$. And let $r_i(t)$ be the number of occurrences of event i in m_t trials, so our likelihood is given by

$$\mathbf{r}(t) \sim Multinomial\,(m_t, \boldsymbol{\mu}(t))$$

that is

$$p(\mathbf{r}(t) \mid \boldsymbol{\mu}_t) = \frac{m_t!}{\prod_{i=0}^{n} r_i(t)!} \prod_{j=0}^{n} \mu_j^{r_j(t)}(t) \tag{4.4.1}$$

subject to the constraint $\sum_{i=0}^{n} r_i(t) = m_t$.

Our prior for $\boldsymbol{\mu}(t)$ is the Dirichlet distribution with parameters $\alpha_0(t), \ldots, \alpha_n(t)$, i.e.

$$p(\boldsymbol{\mu}(t) \mid \boldsymbol{\alpha}(t)) = \frac{\Gamma(\alpha_\bullet(t))}{\prod_{i=0}^{n} \Gamma(\alpha_i(t))} \prod_{j=0}^{n} \mu_j^{\alpha_j(t)-1}(t) \tag{4.4.2}$$

where $\alpha_\bullet = \sum_{i=0}^{n} \alpha_i$.

Before continuing it is worth pointing out one of the major problems that arises when attempting to define any DGLM. In the models of West et. al.(1985) with a univariate parameter it is implicitly assumed that the 'natural' parameter, η, is unconstrained (from the fact that η is simply read off from the exponential family density). This is in fact a neccessary condition, since η satisfies $g(\eta) = \mathbf{F}^T\boldsymbol{\theta}$, and when $\boldsymbol{\theta}$ undergoes the state transformation $\boldsymbol{\theta} \rightarrow \boldsymbol{\theta}' = \mathbf{G}\boldsymbol{\theta} + \mathbf{w}$, η transforms to η' say, where $g(\eta') = \mathbf{F}^T\boldsymbol{\theta}'$. If η is constrained to lie in some set S, bearing in mind that $\boldsymbol{\theta}$ is unconstrained, it is in general not at all clear that η' will also lie in S, as it is required to do. When the underlying parameter is multivariate, as in our case, exactly the same problem exists, but it is now far more likely that this parameter will be constrained. Indeed in our case the parameter $\boldsymbol{\mu}$ is subject to $\sum_{i=0}^{n} \mu_i = 1$ and $\mu_i \in [0,1]$ for all i. One obvious way to overcome this technical difficulty is to require that the transformation from $\boldsymbol{\mu}$ to the 'natural' parameter $\boldsymbol{\eta}$ is such that $\boldsymbol{\eta}$ can take any value in $\mathbb{R}^n$, i.e. $\boldsymbol{\eta}$ is unconstrained. We now construct such a transformation for $\boldsymbol{\mu}$ in the Dirichlet–Multinomial problem above.

Firstly we remove the constraint $\sum_{i=0}^{n} \mu_i = 1$ by dividing through by μ_0. That is, the $n+1$ parameters $\mu_0, \ldots, \mu_n$ are transformed to the n independent paramaters $\lambda_1, \ldots, \lambda_n$ where $\lambda_i = \mu_i/\mu_0$. The only constraint on $\boldsymbol{\lambda}$ is that $\lambda_i > 0$ for $i = 1, \ldots, n$, since given any $\lambda_1, \ldots, \lambda_n$ satisfying this condition, the implied $\boldsymbol{\mu}$ is

$$\mu_0 = \frac{1}{1+\lambda_\bullet} \qquad \mu_i = \frac{\lambda_i}{1+\lambda_\bullet} \qquad\qquad i = 1, \ldots, n$$

where $\lambda_\bullet = \sum_{i=1}^{n} \lambda_i$. Thus the conditions on $\boldsymbol{\mu}$ are satisfied automatically. We see in the next section that this reparameterisation has a natural interpretation in relation to the sequential Gates type model.

It is now clear that our natural parameter $\boldsymbol{\eta}$ can be defined by the transformation

$$\eta_i = h_i(\lambda_i) \qquad\qquad i = 1, \ldots, n$$

where h_i is any invertible function from $(0, \infty)$ to $\mathbb{R}$.

An obvious candidate for such a transform is $h_i(x) = \log x$ for $i = 1, \ldots, n$, since then

$$\eta_i = \log \lambda_i = \log \left(\frac{\mu_i}{1 - \sum_{i=1}^{n} \mu_i} \right) \qquad\qquad i = 1, \ldots, n$$

which is the well known multivariate logistic transform. This has been used in many applications and there is evidence that some people find it easier to think in terms of these log odds than actual probabilities, see Spiegelhalter & Knill-Jones (1986). A similar problem of transforming out constraints is considered in Leonard (1978), where he considers transforming a constrained unknown density, using a ***logistic density transform***, in a density estimation problem.

The distributions of $\boldsymbol{\lambda}$ and $\boldsymbol{\eta}$ are as follows.

$$p(\lambda_1, \ldots, \lambda_n \,|\, \alpha_0, \ldots, \alpha_n) = \frac{\Gamma(\alpha_\bullet)}{\prod_{i=0}^{n} \Gamma(\alpha_i)} \left[\frac{1}{1+\lambda_\bullet} \right]^{\alpha_\bullet} \prod_{i=1}^{n} \lambda_i^{\alpha_i - 1}$$

$$p(\eta_1, \ldots, \eta_n \,|\, \alpha_0, \ldots, \alpha_n) = \frac{\Gamma(\alpha_\bullet)}{\prod_{i=0}^{n} \Gamma(\alpha_i)} \frac{exp(\sum_{i=1}^{n} \eta_i \alpha_i)}{[1 + \sum_{i=1}^{n} exp(\eta_i)]^{\alpha_\bullet}} \qquad (4.4.3)$$

The distribution of $\boldsymbol{\lambda}$ is the standard Inverted Dirichlet distribution. By analogy to the univariate Logistic–Beta distribution we shall refer to (4.4.3) as the Logistic–Dirichlet distribution, i.e. $\boldsymbol{\eta}|\boldsymbol{\alpha} \sim LDi(\boldsymbol{\alpha})$.

Directly from (4.4.3), the moment generating function of $\boldsymbol{\eta}$ is given by

$$\mathbf{E}[\,exp(\sum_{i=1}^{n}\eta_i t_i)\,] = \frac{\Gamma(\alpha_0 - \sum_{i=1}^{n} t_i)\prod_{i=1}^{n}\Gamma(\alpha_i + t_i)}{\prod_{i=1}^{n}\Gamma(\alpha_i)}$$

from which we have

$$\begin{aligned} \mathbf{E}\eta_i &= \gamma(\alpha_i) - \gamma(\alpha_0) \\ Var\,\eta_i &= \dot\gamma(\alpha_i) + \gamma(\alpha_0) \qquad\qquad (4.4.4) \\ Cov\,(\eta_i,\eta_j) &= \dot\gamma(\alpha_0) \qquad\qquad \text{for all } i,j\ i \neq j \end{aligned}$$

where γ is the digamma function defined in chapter 2.

Note that, from the guide relation (4.4.7), these equations impose distributional constraints on the state $\boldsymbol{\theta}$. We shall return to the signifigance of this, and the form of these constraints later in the section.

Returning to (4.4.3), we can simply compute the mode of η_i, $\hat\eta_i$, as

$$\hat\eta_i = \log\left(\frac{\alpha_i}{\alpha_0}\right) \qquad (4.4.5)$$

and the curvature of $p(\boldsymbol{\eta}\,|\,\boldsymbol{\alpha})$ at the mode in the η_i direction as

$$Q_i(\boldsymbol{\eta}) = \frac{\alpha_i(\alpha_* - \alpha_i)}{\alpha_*} \qquad (4.4.6)$$

N.B.

[From now on it is desirable, for clarity, to distinguish between vectors and matrices. Thus, for the rest of this section, I shall adopt the following notation. Column vectors will be denoted by bold characters e.g. $\mathbf{a}$. Matrices will be denoted by underlined bold characters e.g. $\underline{\mathbf{b}}$. A matrix $\underline{\mathbf{b}}$ will be assumed to be made up of column vectors denoted by $\mathbf{b}_i$, that is $\mathbf{b} = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$. If it is neccessary to index a matrix this will be done with a superscript e.g. $\mathbf{b}^k = (\mathbf{b}_1^k, \ldots, \mathbf{b}_n^k)$.]

As in the univariate model we now assume the guide relationship

$$\boldsymbol{\eta}(t) = \mathbf{F}^T(t)\underline{\boldsymbol{\theta}}(t) \qquad (4.4.7)$$

where $\boldsymbol{\theta}(t)$ is the state matrix at time t, and $\mathbf{F}(t)$ is a vector of regressors at time t. Note that (4.4.7) is wholly equivalent to the n univariate guide relationships

$$\eta_i(t) = \mathbf{F}^T(t)\boldsymbol{\theta}_i(t) \qquad\qquad i = 1, \ldots, n$$

$\boldsymbol{\theta}_i(t)$ being the state vector for $\eta_i(t)$.

By analogy with the univariate DGLM our priors are

$$\boldsymbol{\eta}(t-1) \mid D_{t-1} \sim LDi(\boldsymbol{\alpha}(t-1))$$

$$\boldsymbol{\theta}(t-1) \mid D_{t-1} \sim [\mathbf{M}(t-1), \mathbf{C}(t-1)] \tag{4.4.8}$$

where $\mathbf{M}(t-1) = \mathbf{E}[\boldsymbol{\theta}(t-1) \mid D_{t-1}]$ and $\underline{\mathbf{C}}(t-1) = Var[\boldsymbol{\theta}(t-1) \mid D_{t-1}]$. Strictly speaking $\underline{\mathbf{C}}$ is an $n \times n \times n \times n$ matrix, however, for clarity we shall write it as an $n^2 \times n^2$ partitioned matrix as follows

$$\underline{\mathbf{C}} = [\underline{\mathbf{C}}_{ij}] \tag{4.4.9}$$

where $\underline{\mathbf{C}}_{ij} = Cov(\boldsymbol{\theta}_i, \boldsymbol{\theta}_j)$.

Precisely as in the univariate case we model a dynamic development on the state $\boldsymbol{\theta}$ by changing its mean and variance to

$$\boldsymbol{\theta}(t) \mid D_{t-1} \sim [\mathbf{a}(t), \underline{\mathbf{R}}(t)]$$

where

$$\mathbf{a}(t) = \underline{\mathbf{G}}(t)\mathbf{M}(t-1)$$

$$\underline{\mathbf{R}}(t) = [\underline{\mathbf{R}}_{ij}(t)] \quad , \quad \underline{\mathbf{R}}_{ij}(t) = \underline{\mathbf{B}}_{ij}(t)\underline{\mathbf{G}}(t)\underline{\mathbf{C}}_{ij}(t-1)\underline{\mathbf{G}}^T(t)\underline{\mathbf{B}}_{ij}(t) \tag{4.4.10}$$

and $\underline{\mathbf{B}}_{ij}(t)$ is a diagonal matrix of discount factors for $\underline{\mathbf{C}}_{ij}$.

If we now assume $\boldsymbol{\eta}(t) \mid D_{t-1} \sim LDi(\boldsymbol{\alpha}'(t))$, the guide relationship (4.4.7) can be used to deduce values for $\alpha_0'(t), \ldots, \alpha_n'(t)$. A direct extension of the arguments in the univariate case suggest we should obtain these values by solving

$$\mathbf{E}[\boldsymbol{\eta}(t) \mid D_{t-1}] = \mathbf{f}(t) = \mathbf{F}^T(t)\mathbf{a}(t)$$

$$Var[\boldsymbol{\eta}(t) \mid D_{t-1}] = \underline{\mathbf{q}}(t) = [\mathbf{F}^T(t)\underline{\mathbf{R}}_{ij}(t)\mathbf{F}(t)]$$

Before continuing we return to the constraint on $\boldsymbol{\theta}$ imposed by the distributional constraints on $\boldsymbol{\eta}$. This manifests itself above in that we only have $n+1$ parameters $\alpha_0, \ldots, \alpha_n$ with which to reconcile the cumulants of $\boldsymbol{\eta}$ and $\boldsymbol{\theta}$. We clearly have to let the mean of $\boldsymbol{\theta}$ be unconstrained so we are forced to drastically constrain the variance of $\boldsymbol{\theta}$, $\underline{\mathbf{C}}$, or equivalently, only carry a minimal amount of information about its variance. The first obvious constraint on $\underline{\mathbf{C}}$ is as follows

$$\underline{\mathbf{C}}_{ij} = \begin{cases} \underline{\mathbf{C}}_{ii} & i = j \\ \underline{\mathbf{C}}' & i \neq j \end{cases}$$

imposed by the fact that $Cov(\eta_i, \eta_j)$ is the same for all i, j $i \neq j$. Rather than try to specify the other constraints on $\underline{\mathbf{C}}$ in this way our approach will be to isolate what function of $\underline{\mathbf{C}}$ we can specify unconstrained, and then use the constraints to solve for $\mathbf{C}_{ii}$ and $\underline{\mathbf{C}}'$ as functions of this setting. It transpires that such a function is

$$\underline{\mathbf{C}}^* = \sum_{i=1}^{n} \underline{\mathbf{C}}_{ii}$$

We consider the form of $\underline{\mathbf{C}}_{ii}$ and $\underline{\mathbf{C}}'$ as functions of $\underline{\mathbf{C}}^*$ and $\underline{\mathbf{M}}$ in a moment. Given this we can restate our prior on $\boldsymbol{\theta}$ as follows

$$\boldsymbol{\theta}(t-1) \mid D_{t-1} \sim [\underline{\mathbf{M}}(t-1), \underline{\mathbf{C}}^*(t-1)]$$

and if we assume the dynamic development takes $\mathbf{C}^*(t-1)$ to $\mathbf{R}^*(t)$, (4.4.10) becomes

$$\underline{\mathbf{R}}^*(t) = \underline{\mathbf{B}}^*(t)\underline{\mathbf{G}}(t)\underline{\mathbf{C}}^*(t-1)\underline{\mathbf{G}}^T(t)\underline{\mathbf{B}}^*(t)$$

where $\underline{\mathbf{B}}^*(t)$ is a diagonal matrix of discount factors for $\underline{\mathbf{C}}^*(t)$.

Returning to the problem of computing values for $\boldsymbol{\alpha}'(t)$, we can now simply solve

$$\mathbf{E}[\boldsymbol{\eta}(t) \mid D_{t-1}] = \mathbf{f}(t)$$
$$\sum_{i=1}^{n} Var[\eta_i(t) \mid D_{t-1}] = \sum_{i=1}^{n} q_{ii}(t) \equiv \mathbf{F}^T(t)\underline{\mathbf{R}}^*(t)\mathbf{F}(t)$$

If, as in West et. al. (1985), we opt for computational simplicity by replacing $\mathbf{E}[\boldsymbol{\eta}(t) \mid D_{t-1}]$ by *mode* $[\boldsymbol{\eta}(t) \mid D_{t-1}]$, and $Var[\eta_i(t) \mid D_{t-1}]$ by the inverse of the curvature of $p(\boldsymbol{\eta}(t) \mid D_{t-1})$ at the mode in the η_i direction, we get, from (4.4.5) and (4.4.6),

$$\alpha_0'(t) = \frac{1 + \sum_{i=1}^{n} e^{f_i(t)}}{\sum_{j=1}^{n} q_{jj}(t)} \sum_{i=1}^{n} \frac{e^{-f_i(t)}}{1 - e^{f_i(t)} + \sum_{j=1}^{n} e^{f_j(t)}}$$
$$\alpha_i'(t) = \alpha_0'(t) e^{f_i(t)} \tag{4.4.11}$$

A pleasing property of this updating is as follows. If $\mathbf{G}(t)$ is the identity matrix, (i.e. we are assuming no growth trend), and the regressors and seasonal effects on two succesive contracts are the same, then a very acceptable way to model a dynamic development *directly* on $\boldsymbol{\eta}(t)$ would be to use the Simple Power Steady model. As discussed in §4.2 this would result in the updating

$$\boldsymbol{\alpha}'(t) = k_t \boldsymbol{\alpha}(t-1)$$

where $k_t \in (0,1]$ is the discount factor. It is easy to show that the updating in (4.4.11) is equivalent to this updating with

$$k_t = \frac{\sum_{i=1}^{n} Var\,(\eta_i(t-1)\,|\,D_{t-1})}{\sum_{i=1}^{n} Var\,(\eta_i(t)\,|\,D_{t-1})}$$

$$= \frac{\mathbf{F}^T(t)\underline{\mathbf{C}}^*(t-1)\mathbf{F}(t)}{\mathbf{F}^T(t)\mathbf{B}^*(t)\underline{\mathbf{C}}^*(t-1)\mathbf{B}^*(t)\mathbf{F}(t)}$$

So the simple power steady model evolution is a just special case of the evolution specified in (4.4.11).

In the example in the next section it is found that approximating the variance by the inverse of the curvature can cause problems, especially if $e^{f_1},\ldots,e^{f_n}$ are large. We can overcome this problem by solving the following equation directly for α_0'.

$$f(\alpha_0') = \dot\gamma(\alpha_0') + \sum_{i=1}^{n}\dot\gamma(e^{f_i}\alpha_0') - \sum_{i=1}^{n} q_{1i} = 0$$

The other α' values are then as in (4.4.11). A direct search method on $f(.)$ can easily be constructed by noting that $f(.)$ is a decreasing function (such a method is described in the next section). The above comments about the steady model being a special case still apply, however k_t will now be given by

$$k_t = \frac{\alpha_0'(t)}{\alpha_0(t)}$$

We now return to the problem of specifying $\underline{\mathbf{C}}_{*i}$ and $\underline{\mathbf{C}}'$ as functions of $\underline{\mathbf{C}}^*$, $\underline{\mathbf{M}}$ and the vector $\underline{\mathbf{F}}$ which specifies the guide relation. The guide relation (4.4.7) immediately gives us

$$\underline{\mathbf{C}}_{*i} = \frac{Var\,\eta_i}{\sum_{j=1}^{n} Var\,\eta_j}\underline{\mathbf{C}}^*$$
$$\underline{\mathbf{C}}' = \frac{Cov(\eta_i,\eta_j)}{\sum_{j=1}^{n} Var\,\eta_j}\underline{\mathbf{C}}^*$$

Of course given $\mathbf{M}$, $\underline{\mathbf{C}}^*$ and $\mathbf{F}$ we can determine values for $\alpha_0,\ldots,\alpha_n$, from (4.4.11), and hence values for the cumulants of η in the above. If we once again replace the mean by the mode and the variance by the inverse of the curvature the above become

$$\underline{\mathbf{C}}_{*i} = \left(\frac{e^{-f_i}[1-e^{f_i}+\sum_{j=1}^{n} e^{f_j}]^{-1}}{\sum_{k=1}^{n} e^{-f_k}[1-e^{f_k}+\sum_{j=1}^{n} e^{f_j}]^{-1}}\right)\underline{\mathbf{C}}^* \equiv \frac{b_i}{b_*}\underline{\mathbf{C}}^* \qquad (4.4.12)$$

$$\underline{\mathbf{C}}' = \left(\frac{n+1}{2n\sum_{k=1}^{n} e^{-f_k}[1-e^{f_k}+\sum_{j=1}^{n} e^{f_j}]^{-1}}\right)\underline{\mathbf{C}}^* \equiv \frac{n+1}{2nb_*}\underline{\mathbf{C}}^*$$

where $f_i = (\mathbf{F}^T\underline{\mathbf{M}})_i$, $b_i = e^{-f_i}[1-e^{f_i}+\sum_{j=1}^{n} e^{f_j}]^{-1}$ and the approximation $\dot\gamma(\alpha_i)+\dot\gamma(\alpha_0) = \alpha_*/\alpha_i(\alpha_* - \alpha_i)$ gives $\dot\gamma(\alpha_0) = (n+1)/2n\alpha_0$ when $\alpha_i = \alpha_0$ for $i=1,\ldots,n$.

Given the observations $r_0(t), \ldots, r_n(t)$, the conjugacy of the multinomial and Dirichlet distributions gives the updating

$$\boldsymbol{\eta}(t) \mid D_t \sim LDi(\alpha_0(t), \ldots, \alpha_n(t))$$

$$\alpha_i(t) = \alpha_i'(t) + r_i(t)$$

In order to update our beliefs about $\boldsymbol{\theta}(t)|D_t$, i.e. obtain $\mathbf{M}(t)$ and $\mathbf{C}^*(t)$, we first assume that $\underline{\mathbf{C}}(t)$ is unconstrained and proceed as in West et. al. (1985). This leads to the updating in (4.4.14), which would be applicable if $\mathbf{C}$ were unconstrained. However when the constrained values are substituted in, (4.4.14) reduces to the very pleasing updating in (4.4.15).

First note the identities

$$\underline{\mathbf{M}}(t) = \mathbf{E}_{\boldsymbol{\eta}}\mathbf{E}[\boldsymbol{\theta}(t) \mid \boldsymbol{\eta}(t), D_t]$$

$$\underline{\mathbf{C}}_{ij}(t) = \mathbf{E}_{\boldsymbol{\eta}}[Cov\,(\boldsymbol{\theta}_i(t), \boldsymbol{\theta}_j(t) \mid \boldsymbol{\eta}(t), D_t)] + Cov_{\boldsymbol{\eta}}(\mathbf{E}[\boldsymbol{\theta}_i(t) \mid \boldsymbol{\eta}(t), D_t], \mathbf{E}[\boldsymbol{\theta}_j(t) \mid \boldsymbol{\eta}(t), D_t]) \tag{4.4.13}$$

Precisely as in the univariate case $\boldsymbol{\eta}(t)$ is sufficent for $\underline{\boldsymbol{\theta}}(t)$ i.e.

$$p(\underline{\boldsymbol{\theta}}(t) \mid \boldsymbol{\eta}(t), D_t) = p(\underline{\boldsymbol{\theta}}(t) \mid \boldsymbol{\eta}(t), D_{t-1})$$

Thus if we now construct an estimator of $\mathbf{E}[\boldsymbol{\theta}(t) \mid \boldsymbol{\eta}(t), D_{t-1}]$ we can feed it in to (4.4.13) in place of $\mathbf{E}[\boldsymbol{\theta}(t) \mid \boldsymbol{\eta}(t), D_t]$ to produce our updated cumulants for $\boldsymbol{\theta}(t) \mid D_t$. Take this estimator to be of the form

$$\mathbf{E}[\underline{\boldsymbol{\theta}}(t) \mid \boldsymbol{\eta}(t), D_{t-1}] = \underline{\mathbf{d}}^0(t) + \sum_{i=1}^{n} \mathbf{d}^i(t)\eta_i(t) = \underline{\mathbf{d}}(t)$$

where $\underline{\mathbf{d}}^0, \ldots, \mathbf{d}^n$ are chosen to minimise

$$\sum_{i=1}^{n} Trace\,\mathbf{E}_{\boldsymbol{\eta}}\mathbf{E}[(\boldsymbol{\theta}_i - \mathbf{d}_i)(\boldsymbol{\theta}_i - \mathbf{d}_i)^T \mid \boldsymbol{\eta}(t), D_{t-1}]$$

If this minimum is achieved at $\underline{\mathbf{d}}$ it is natural to take

$$Cov\,(\boldsymbol{\theta}_i(t), \boldsymbol{\theta}_j(t) \mid \boldsymbol{\eta}(t), D_{t-1}) = \mathbf{E}_{\boldsymbol{\eta}}\mathbf{E}[(\boldsymbol{\theta}_i(t) - \hat{\mathbf{d}}_i)(\boldsymbol{\theta}_j(t) - \hat{\mathbf{d}}_j)^T \mid \boldsymbol{\eta}(t), D_{t-1}]$$

Performing this minimisation yields

$$\begin{aligned}\underline{\mathbf{d}} &= (\mathbf{a} - \sum_{k=1}^{n} f_k\mathbf{s}^k\mathbf{q}^{-1}) + \sum_{k=1}^{n} \eta_k\mathbf{s}^k\mathbf{q}^{-1} \\ &= \mathbf{a} + \sum_{k=1}^{n}(\eta_k - f_k)\mathbf{s}^k\mathbf{q}^{-1}\end{aligned}$$

where $\mathbf{s}^k(t) = Cov(\boldsymbol{\theta}(t), \eta_k(t) \mid D_{t-1}) = (\mathbf{s}_1^k(t), \ldots, \mathbf{s}_n^k(t))$, $\mathbf{s}_j^k(t) = cov(\boldsymbol{\theta}_j(t), \eta_k(t) \mid D_{t-1}) = \mathbf{R}_{k,j}(t)\mathbf{F}(t)$. And hence

$$Cov(\boldsymbol{\theta}_i(t), \boldsymbol{\theta}_j(t) \mid \boldsymbol{\eta}(t), D_{t-1}) = \underline{\mathbf{R}}_{ij} - \sum_{k=1}^{n} [(\mathbf{s}^k \underline{\mathbf{q}}^{-1})_i (\mathbf{s}_j^k)^T + \mathbf{s}_i^k (\mathbf{s}^k \underline{\mathbf{q}}^{-1})_j^T] + \sum_{k=1}^{n}\sum_{l=1}^{n} q_{kl} (\mathbf{s}^k \mathbf{q}^{-1})_i (\mathbf{s}^l \mathbf{q}^{-1})_j^T$$

Substituting these in to (4.4.13) yields

$$\mathbf{M}(t) = \mathbf{a} + \sum_{k=1}^{n} (g_k - f_k) \mathbf{s}^k \underline{\mathbf{q}}^{-1}$$

$$\underline{\mathbf{C}}_{ij}(t) = \underline{\mathbf{R}}_{ij} - \sum_{k=1}^{n} [(\mathbf{s}^k \underline{\mathbf{q}}^{-1})_i (\mathbf{s}_j^k)^T + \mathbf{s}_i^k (\mathbf{s}^k \underline{\mathbf{q}}^{-1})_j^T] + \sum_{k=1}^{n}\sum_{l=1}^{n} (p_{kl} + q_{kl}) (\mathbf{s}^k \mathbf{q}^{-1})_i (\mathbf{s}^l \mathbf{q}^{-1})_j^T \tag{4.4.14}$$

where all the quantities on the right are functions of t, and $g_k(t) = \mathbf{E}[\eta_k(t) \mid D_t]$ and $p_{kl}(t) = Cov(\eta_k(t), \eta_l(t) \mid D_t)$.

Equation (4.4.14) would be our final updating if $\mathbf{C}$ were unconstrained. However we can simplify these updating equations using the structure we have on $\mathbf{R}_{ij}$, that is

$$\underline{\mathbf{R}}_{ii} = \frac{b_i}{b_\bullet} \underline{\mathbf{R}}^*$$
$$\underline{\mathbf{R}}' = \frac{c}{b_\bullet} \underline{\mathbf{R}}^*$$

where, if we use the previous approximations, $b_i = e^{-f_i}[1 - e^{f_i} + \sum_{j=1}^{n} e^{f_j}]^{-1}$, $c = (n+1)/2n$ and $f_i = (\mathbf{F}^T\mathbf{a})_i$. Note that the above equations hold when we are not using these approximations, the only difference being that the forms of b_i and c will be different.

This immediately gives

$$\underline{\mathbf{q}} = \frac{\mathbf{F}^T \underline{\mathbf{R}}^* \mathbf{F}}{b_\bullet} [diag(b_i - c) + c\mathbf{e}\mathbf{e}^T]$$

$$\mathbf{s}_j^k \equiv \underline{\mathbf{R}}_{kj} \mathbf{F} = \begin{cases} \frac{b_k}{b_\bullet} \underline{\mathbf{R}}^* \mathbf{F} & j = k \\ \frac{c}{b_\bullet} \underline{\mathbf{R}}^* \mathbf{F} & j \neq k \end{cases}$$

where $diag$ denotes a diagonal matrix and $\mathbf{e}$ is the $n \times 1$ vector of ones. A little manipulation of these yields

$$(\mathbf{s}^k \underline{\mathbf{q}}^{-1})_i = \begin{cases} 0 & i \neq k \\ \frac{\mathbf{R}^* \mathbf{F}}{\mathbf{F}^T \mathbf{R}^* \mathbf{F}} & i = k \end{cases}$$

whence the updating (4.4.14) becomes

$$\mathbf{M}_i(t) = \mathbf{a}_i + \frac{(g_i - f_i)\underline{\mathbf{R}}^* \mathbf{F}}{\mathbf{F}^T \mathbf{R}^* \mathbf{F}}$$

$$\underline{\mathbf{C}}^*(t) = \underline{\mathbf{R}}^* - \frac{(\underline{\mathbf{R}}^* \mathbf{F})(\underline{\mathbf{R}}^* \mathbf{F})^T}{\mathbf{F}^T \underline{\mathbf{R}}^* \mathbf{F}} \left[1 - \frac{\sum_{i=1}^{n} p_{ii}}{\mathbf{F}^T \underline{\mathbf{R}}^* \mathbf{F}} \right] \tag{4.4.15}$$

where once again all the quantities on the right are functions of t. These equations are very appealing as they are directly analogous to the Beta–Binomial updating equations given in chapter 2.

The full system of updating parameters is shown in figure (4.4.16).

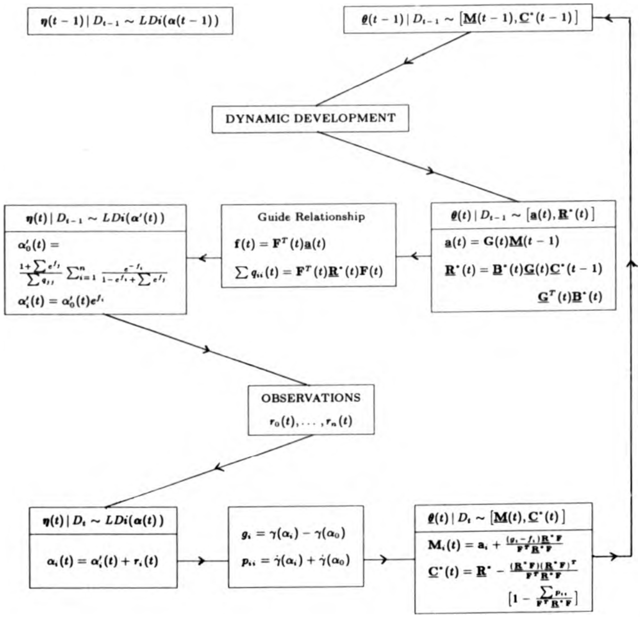

$\eta(t-1) \mid D_{t-1} \sim LDi(\alpha(t-1))$
$\theta(t-1) \mid D_{t-1} \sim [\underline{\mathbf{M}}(t-1), \underline{\mathbf{C}}^*(t-1)]$
DYNAMIC DEVELOPMENT
$\eta(t) \mid D_{t-1} \sim LDi(\alpha'(t))$
$\alpha'_0(t) =$
$\frac{1+\sum e^{f_j}}{\sum q_{jj}} \sum_{i=1}^{n} \frac{e^{-f_i}}{1-e^{f_i}+\sum e^{f_j}}$
$\alpha'_i(t) = \alpha'_0(t) e^{f_i}$
Guide Relationship
$\mathbf{f}(t) = \mathbf{F}^T(t)\underline{\mathbf{a}}(t)$
$\sum q_{ii}(t) = \mathbf{F}^T(t)\underline{\mathbf{R}}^*(t)\mathbf{F}(t)$
$\theta(t) \mid D_{t-1} \sim [\underline{\mathbf{a}}(t), \underline{\mathbf{R}}^*(t)]$
$\underline{\mathbf{a}}(t) = \mathbf{G}(t)\underline{\mathbf{M}}(t-1)$
$\underline{\mathbf{R}}^*(t) = \underline{\mathbf{B}}^*(t)\underline{\mathbf{G}}(t)\underline{\mathbf{C}}^*(t-1)$
$\underline{\mathbf{G}}^T(t)\underline{\mathbf{B}}^*(t)$
OBSERVATIONS
$r_0(t), \ldots, r_n(t)$
$\eta(t) \mid D_t \sim LDi(\alpha(t))$
$\alpha_i(t) = \alpha'_i(t) + r_i(t)$
$g_i = \gamma(\alpha_i) - \gamma(\alpha_0)$
$p_{ii} = \dot{\gamma}(\alpha_i) + \dot{\gamma}(\alpha_0)$
$\theta(t) \mid D_t \sim [\underline{\mathbf{M}}(t), \underline{\mathbf{C}}^*(t)]$
$\mathbf{M}_i(t) = \mathbf{a}_i + \frac{(g_i - f_i)\underline{\mathbf{R}}^*\mathbf{F}}{\mathbf{F}^T\underline{\mathbf{R}}^*\mathbf{F}}$
$\underline{\mathbf{C}}^*(t) = \underline{\mathbf{R}}^* - \frac{(\underline{\mathbf{R}}^*\mathbf{F})(\underline{\mathbf{R}}^*\mathbf{F})^T}{\mathbf{F}^T\underline{\mathbf{R}}^*\mathbf{F}}$
$[1 - \frac{\sum p_{ii}}{\mathbf{F}^T\underline{\mathbf{R}}^*\mathbf{F}}]$

FIGURE 4.4.16

4.5 A DGLM For The Sequential Gates Type Model.

In this section we consider the practical problems faced in applying the DGLM of §4.4 to the sequential Gates type model of §4.1.

As in §4.1 to make the DGLM useful we must allow all the parameters in the model to become functions of our chosen markup m, i.e. $\boldsymbol{\alpha}(t) = \boldsymbol{\alpha}(t,m)$ and $\boldsymbol{\theta}(t) = \boldsymbol{\theta}(t,m)$. To fully update our parameters we now require the lowest markup made by our competitors, $m^*(t)$, as well as $\mathbf{r}(t)$, where $\mathbf{r}(t)$ now indicates the identity of the competitor using the markup $m^*(t)$. Given this, the updating of the distribution of $\boldsymbol{\eta}(t,m) \mid D_{t-1}$ to that of $\boldsymbol{\eta}(t,m) \mid D_t$ is given by

$$\left.\begin{aligned} \alpha_0(t,m) &= \alpha_0'(t,m) + 1 \\ \alpha_i(t,m) &= \alpha_i'(t,m) \end{aligned}\right\} \qquad m < m^*$$

$$\left.\begin{aligned} \alpha_0(t,m) &= \alpha_0'(t,m) \\ \alpha_i(t,m) &= \alpha_i'(t,m) + r_i(t) \end{aligned}\right\} \qquad m > m^*$$

Which are directly analogous to the updating equations in §4.1.

The quantity we shall use in computing our optimal markup is $\mathbf{E}[\mu_0(t,m) \mid D_{t-1}] \equiv \hat{p}_0(t,m)$, since it is the natural estimator of $p_0(t,m) = P(\text{we win contract at time } t \text{ with markup } m)$. $\hat{p}_0(t,m)$ is simply given by

$$\hat{p}_0(t,m) = \frac{\alpha_0'(t,m)}{\alpha_\bullet'(t,m)}$$

We look at computing the optimal markup from this later in the section, but first we discuss one of the practical advantages of using the Dirichlet–Multinomial model.

It could be asked why we should use the Dirichlet–Multinomial model in preference to the simpler Beta–Binomial model. The Beta–Binomial model would require only one state vector for all of our competitors, and the specification of only two alpha parameters α_0, α_1, which would be set by fixing the mean and variance of $F^T\boldsymbol{\theta}$. This could be described as modelling the problem as 'us' versus 'the rest'. At first glance this does not appear to disadvantageous since in the Dirichlet–Multinomial model we also only have one α parameter with which to specify the variance structure. The first response to this is that the Dirichlet–Multinomial model allows $n-1$ parameters to fix means separately for each competitor. But perhaps a more compelling argument in favour of the Dirichlet–Multinomial model is that it allows far greater flexibility in intervention, as we now discuss.

Harrison (1988) stresses the principle of 'management by exception' in relation to forecasting models such as the DGLM of the last section. By this he means that the output from the model should be combined with any exceptional subjective information the operator may have at a given time period. That is, the operator should feel free to intervene in the free running of the model. The problem we consider now is how to simply incorporate subjective information by adjusting the parameters in the model. Note that we might intervene for two possible reasons. The first arises if the operator learns of some change in the environment which requires permanent readjustment of the model. In this case once the parameters have been altered the model will be left to run starting from these new values. The second arises if the operator knows that some unusual circumstances pertain only at a given time period. Here the parameters will be adjusted in the same way for decision making at the given time period. However the parameters will not then be updated and the model will run on with its old parameter values.

In the DGLM of §4.4 the obvious place to intervene is when the parameters $\alpha'_0, \ldots, \alpha'_n$ are specified. The form of this intervention is based on the following two familiar facts

(1) $\frac{\alpha'_i(m)}{\alpha'_\bullet(m)} = P(F_i$ wins the forthcoming contract $|$ we use markup $m)$

(2) $\alpha'_\bullet(m)$ is a measure of our confidence in the models parameter values

We now consider some general situations where intervention may be called for, and the form this intervention should take.

(i) Non–specific changes in whole bidding environment.

These can arise due to changes in raw material prices or supplies say, which could, unknown to us, be beneficial or detrimental to a given competitor depending on their current circumstances. This sort of change is usually only significant in that it makes us less sure how our competitors are likely to behave, that is we have less confidence in our current parameter settings. We can simply incorporate this by reducing the value of $\alpha'_\bullet(m)$ and reseting $\alpha'_i(m)$ to leave $\alpha'_i(m)/\alpha'_\bullet(m)$ unchanged. This reduced value of $\alpha'_\bullet(m)$ makes the model more sensitive to the new data, and thus allows it to rapidly readjust itself to the new environment.

(ii) Individual company becomes more/less competitive.

This can arise if we learn one of our competitors is in financial difficulty, in which case desperation for work is likely to make them more competitive. Or alternatively they have plenty of work in which case they are likely to be less competitive. The way to approach this is simply to increase (F_i more competitive) or decrease (F_i less competitive) the parameter $\alpha_i'(m)$. This will of course alter $\alpha_0'(m)$, so, depending on our confidence in our information about F_i, we may wish to readjust $\alpha_0'(m)$ as (i).

(iii) Specific changes in whole bidding environment.

If we learn something specific about our competitors, for example they are all likely to start using smaller/larger markups, then it clearly could be costly to simply wait for the model to readjust as in (i). We can handle this immediately by noting that if all our competitors start using smaller/larger markups this is equivalent to our company becoming less/more competitive. So we can simply readjust our parameter $\alpha_0'(m)$ as in (ii).

(iv) 'One–off' contracts.

All the above have been examples of permanent readjustments to the running of the model. A case where we might only want to adjust our parameters temporarily is as follows. We learn that one of our competitors F_i is either in no position to compete for, or has no chance of winning, a given contract. Possibly due to non–price considerations. This will only be beneficial if all our competitors are privy to this information, whence we set $\alpha_i'(m) = 0$ in the calculation of our markup for this particular contract. Our own optimal markup will consequently be larger.

We now consider the model specifications, that is the definition of states, and the prior settings.

Consider first the definition of our state matrix $\theta(t,m)$. Note that the i^{th} column of θ is the state vector for company i, thus for the definition of the states and the prior settings we shall consider just one company and drop the subscript i. One then simply repeats the procedure for each company. Loosely speaking the states for a particular company fall in to three categories, namely

(1) seasonal effects

(2) regressor effects

(3) growth effects

A growth effect is not likely to be relevant as markup policy is unlikely to drift in a systematic way. We are more likely to see discrete jumps due to changes in our competitors policy, these are best handled by intervention as we have just discussed.

Examples of possible regressors are given below.

c_t *Cost of contract at time t*. This will have a positive effect (or state), i.e. the higher the cost of the contract the less likely we are to win with a given markup, since firms are likely to be prepared to take smaller markups for larger contracts.

π_t *Prestige of contract at time t*. Defined on a scale $(0, 1)$ say. Like c_t this will have a positive effect, since firms will bid more competitively for high prestige contracts.

d_t *Unseasonal demand for work at time t*. As with c_t and π_t we would expect this to have a positive effect.

Bearing in mind that our observations will be arriving at non regular time intervals we can specify a monthly seasonal effect by fitting a term such as

$$\theta(t, m)\sin\left(\frac{\pi(s_t - 1)}{12}\right) + \theta'(t, m)\cos\left(\frac{\pi(s_t - 1)}{12}\right) \tag{4.5.1}$$

where $s_t \in \{1, 2, \ldots, 12\}$ is the month at the time of the t^{th} contract ($s_t = 1$ is January say). A first harmonic term such as (4.5.1) will often suffice since seasonal effects are likely to be due to factors such as tempereature and number of daylight hours.

Summarising, our proposed marginal model for the behaviour of a single company is

$$\eta(t, m) = \theta_0(t, m) + \theta_1(t, m)\sin\left(\frac{\pi(s_t - 1)}{12}\right) + \theta_1'(t, m)\cos\left(\frac{\pi(s_t - 1)}{12}\right) + \sum_{i=2}^{l} \theta_i(t, m)\rho_i(t) \tag{4.5.2}$$

where

$$\begin{aligned} \theta_0(t, m) &= \text{ basic setting} \\ \theta_1'(t, m) \text{ and } \theta_1(t, m) &= \text{ seasonal effects at time } t \\ \rho_i(t) &= \text{ value of regressor } \rho_i \text{ at time } t \\ \theta_i(t, m) &= \text{ effect of regressor } \rho_i \text{ at time } t \end{aligned}$$

which in the terminology of the DGLM is equivalent to

$$\mathbf{F}(t) = \left(1, \sin\left(\frac{\pi(s_t - 1)}{12}\right), \cos\left(\frac{\pi(s_t - 1)}{12}\right), \rho_2(t), \ldots, \rho_l(t)\right)^T$$

and, since we have no growth terms

$$\underline{\mathbf{G}}(t) = \mathbf{I}_{l \times l} \qquad \text{for all } t$$

the $l \times l$ identity matrix.

Consider now our prior settings $\mathbf{M}(0,m) = \mathbf{E}[\theta(0,m)]$ and $\mathbf{C}(0,m) = Cov(\theta(0,m))$. Note that if we are working on the settings for company i $\theta(t,m)$ will be the i^{th} column of the state matrix $\theta(t,m)$ and the expectation and covariance above correspond to $\mathbf{M}_i(0,m)$ and $\mathbf{C}_{ii}(0,m)$. For simplicity we shall consider only uninformative prior settings. Obviously other prior beliefs can be incorporated in the same manner.

Clearly for all effects other than θ_0 it is reasonable as an uninformative setting to take $\mathbf{E}[\theta_i(0,m)] = k_i$, that is a constant effect for all m. However it is then clearly fallacious to take $\mathbf{E}[\theta_0(0,m)]$ =constant, since this would imply $\mathbf{E}[\eta(0,m)]$ =constant which is clearly not so. In order to set $\mathbf{E}[\theta_0(0,m)]$ we shall construct a plausible form for $\mathbf{E}[\eta(0,m)]$ in the absence of all effects and take this as our expectation of $\theta_0(0,m)$. Recall the definition of $\eta(t,m)$ is

$$\eta(t,m) = \log\left(\frac{p(t,m)}{p_0(t,m)}\right)$$

$$p(t,m) = P(\text{competitor wins forthcoming contract }|\text{we use markup } m\ )$$

$$p_0(t,m) = P(\text{we win forthcoming contract with markup } m\ )$$

Firstly restrict the values m can take to a realistic range, $m \in (L,U)$ say. Thus, from the definition of η above, $\eta(0,m)$ is an increasing function of m on (L,U). It will prove very convenient for the computation of our optimal markup (discussed later) if our prior expectation is a step function of m. A simple example is as follows. If we specify the values $\bar{\eta}_L = \mathbf{E}[\eta(0,L)]$ and $\bar{\eta}_U = \mathbf{E}[\eta(0,U)]$, then given a refinement parameter ν we could take

$$\mathbf{E}[\eta(0,m)] = \mathbf{E}[\theta_0(0,m)] = \frac{\bar{\eta}_U - \bar{\eta}_L}{U-L}\left[L + \frac{U-L}{\nu}(i+\frac{1}{2})\right] + \frac{U\bar{\eta}_L - L\bar{\eta}_U}{U-L} \tag{4.5.3}$$
$$m \in (L + \frac{i(U-L)}{\nu}, L + \frac{(i+1)(U-L)}{\nu}] \quad i = 0,\ldots,\nu-1$$

For example when $\nu = 10$, $(L,U) = (1,1.5)$, $\bar{\eta}_U = 2$ and $\bar{\eta}_L = -2$ this appears as in figure (4.5.4) on p.60.

This piecewise form is nothing like as cumbersome as it seems since the moment we start receiving data all our means and variances will be defined as piecewise functions of m. So we have to be prepared to handle this form anyway. To set $\bar{\eta}_U$ and $\bar{\eta}_L$ we simple estimate how many times more/less likely than us the given company is to win win a contract when we use a markup near U/L. For example we might think that the company is ten times more likely than us to win the contract if we use a markup near U, whence $p/p_0 = 10$ and $\bar{\eta}_U = \log 10$.

It seems very reasonable to assume all effects are apriori independent, thus $\mathbf{C}^*(0,m)$ will be a diagonal matrix for all m. The procedure for setting $k_1, \ldots, k_l$ and $Var(\theta_0(0,m))$, $\ldots, Var(\theta_l(0,m))$ are similar. For regressor effects proceed as follows. Consider a change in regressor from r to r' (around the 'average' value of the regressor) with other regressors unchanged, and estimate the corresponding change in η, $\eta \to \eta'$ say, and also the range of realistic possibles values of η', $\eta' \in [\eta' - \epsilon_1, \eta' + \epsilon_2]$ say. We can then take

$$k = \frac{\eta - \eta'}{r - r'}$$

$$Var(\theta(0,m)) = \left[\frac{\epsilon_1 + \epsilon_2}{6(r - r')}\right]^2$$

The form for the variance is based on the fact that we hope the distribution of θ will not be too far from being normal. And if it were exactly normal we could say that a realistic range for θ would be $\mathbf{E}[\theta] \pm 3\sqrt{var\,\theta}$, which can be rearranged to give $var\,\theta = [\text{realistic range of } \theta/6]^2$.

For the seasonal effect we can proceed similarly to obtain the mean and variance of the amplitude of the combined sin and cos terms. It is however also necessary to set a phase by deciding how the effect should be distributed between the sin term and the cos term. It may for example not be unreasonable to apriori take the mean of the cos term to be zero. This would be equivalent to saying that January is the month when competition for contracts is toughest and July the month when competition is weakest.

To set the variance of $\theta_0(0,m)$ we simply need to specify a range of values on $\eta(0,m)$ for each piecewise segment and apply the above approximation, i.e. $Var(\theta_0(0,m)) = [\text{range}/6]^2$.

In order to ensure that we do not arrive at too small a variance we must remember that the range we specify must contain almost every conceivable value η' can take.

In the long term, when we have a reasonable amount of data, these prior settings will have little effect on our decisions. So our main aim when setting prior parameters is simply to get values which are of the right magnitude and portray obvious apriori facts. The above procedures provide a method for satisfying these aims.

Our final task is to compute our optimal markup for a given contract. In the following example we will simply choose our markup to maximise our expected percentage profit function, $prof(t,m)$, on the next contract, where

$$prof(t,m) = (m-1)\hat{p}_0(t,m) = \frac{(m-1)\alpha_0'(t,m)}{\alpha_*'(t,m)}$$

Our step prior and the form of the updating guarantees that *prof* will always be a step function of m. Thus the only markups we need consider are those at the the upper ends of the prior ranges and each subsequent lowest markup made by our competitors. There is now no problem in computing prof at each of these markups and choosing the markup which yields the highest profit.

All the comments in §4.3 concerning more realistic utilities, by for example taking in to account the amount of work we are currently engaged on, also apply here. The above comments about the step function remaining over time will of course also remain true, so these more general utilities will not make computation of the optimal markup any more complex.

Before giving an example it is worth mentioning a couple of the computational techniques used in the example. Consider first computation of the digamma function (γ) and the trigamma function ($\dot\gamma$). The procedures found to work well in the example are as follows. First note the following *exact* formulae (Abramowitz & Stegun (1964) pp 258–260):

$$\gamma(n) = -\gamma + \sum_{i=1}^{n-1} \frac{1}{i} \qquad n \in \mathbb{N}$$

$$\gamma(n+\frac{1}{2}) = -\gamma - 2\log 2 + 2\sum_{i=1}^{n} \frac{1}{2i-1} \qquad n \in \mathbb{N}$$

$$\dot\gamma(n) = \frac{\pi^2}{6} - \sum_{i=1}^{n-1} \frac{1}{i^2} \qquad n \in \mathbb{N}$$

$$\dot\gamma(n+\frac{1}{2}) = \frac{\pi^2}{2} - 4\sum_{i=1}^{n} \frac{1}{(2i-1)^2} \qquad n \in \mathbb{N}$$

Where $\gamma \simeq -.577216$ is Eulers gamma.

For $x \geq 2$ compute $\gamma(\text{int}(x))$, $\gamma(\text{int}(x)+\frac{1}{2})$, $\gamma(\text{int}(x)+1)$, using the above. A value for $\gamma(x)$ can then be obtained by quadratic interpolation on these three points. For $x < 2$ the above procedure is unsatisfactory, so note the following series expansions

$$\gamma(1+x) = -\gamma + \sum_{i=1}^{\infty} \frac{x}{i(i+x)} \qquad x \neq -1, -2, \ldots$$

$$\dot\gamma(x) = \sum_{i=0}^{\infty} \frac{1}{(x+i)^2} \qquad x \neq 0, -1, \ldots$$

γ and $\dot\gamma$ could be computed by taking a large number of terms in these series. However this is somewhat slow to do every time we want to compute values. A quicker method is simply to compute $\gamma(x)$ and $\dot\gamma(x)$ at say 20 points in the range $(0,2)$ using the series expansion and compute other values by piecewise linear interpolation.

Finally consider solving numerically the equation

$$f(x) = \bar{\gamma}(x) + \sum_{i=1}^{n} \dot{\gamma}(e^{J_i} x) - \sum_{i=1}^{n} q_i = 0$$

It is clear that for $x > 0$ $f(x)$ is a decreasing function. So it is easy to use a method that searches directly on $f(x)$. This is also very desirable since it avoids the added complexity of computing γ^{-1} or $\bar{\gamma}$. The method used in the example is simply a modified midpoint method. Given the starting value x_0, taken as the value given by (4.4.11), compute $f(x_0)$. If $f(x_0) > 0$ take $x_0' = 1.5x_0$ otherwise take $x_0' = \frac{1}{1.5}x_0$. Let A be given by

$$A = x_0 - \frac{f(x_0)\,[x_0 - x_0']}{f(x_0) - f(x_0')}$$

and let $h = f(A)$. A is the point where the straight line through $(x_0, f(x_0))$ and $(x_0', f(x_0'))$ crosses the x axes.

Set x_0 to be x_0' if $f(x_0') < f(x_0)$, otherwise leave it unchanged. Set x_0' to be A. Recompute A and repeat the procedure until h is as near 0 as required. The required value of α_0' is then the corresponding value of A. This procedure will always converge, and in practice has rarely taken more than 2 or 3 iterations.

Example.

We are to compete against three companies ($n = 3$) on a series of contracts. We shall model a seasonal effect and regress on the prestige of the contract at time t, π_t. There is overwhelming evidence, in this bidding environment, that January is the month when competition for contracts is toughest. Consequently, to reduce on computation, we shall model the seasonal effect by taking just the sin component of (4.5.1). Thus our model parameters are as follows

$$\mathbf{F}(t) = \left(1, \sin\left(\frac{\pi(s_t - 1)}{12}\right), \pi_t\right)$$

$$\underline{\theta}(t,m) = \begin{pmatrix} \theta_0^1(t,m) & \theta_0^2(t,m) & \theta_0^3(t,m) \\ \theta_1^1(t,m) & \theta_1^2(t,m) & \theta_1^3(t,m) \\ \theta_2^1(t,m) & \theta_2^2(t,m) & \theta_2^3(t,m) \end{pmatrix}$$

Apriori we shall assume our competitors are identical and thus consider prior settings for just one company, i.e. just one column of θ.

Consider first $\theta_2^i(0,m)$ the effect of the prestige. Assume for a contract of prestige .2 we believe our chances of winning, with a given markup, to be twice that of company i, thus

$\eta_i(0,m) = \log \frac{1}{2}$. If we now learn that the prestige is in fact .8 we might feel that we are only $1\frac{1}{2}$ times more likely to win than F_i but certainly in the range .5 to 2.5 times as likely, suggesting $\eta_i'(0,m) = \log \frac{2}{3}$ but $\eta_i'(0,m) \in [\log \frac{2}{5}, \log 2]$. These imply

$$\mathbf{E}[\theta_2^*(0,m)] = .48 \qquad \text{for all } i \text{ and } m$$

$$Var[\theta_2^*(0,m)] = .20 \qquad \text{for all } i \text{ and } m$$

Similarly for the seasonal effect we may feel that our chances of winning, with a given markup, in July to be twice that of F_i and that this would fall to $1\frac{1}{3}$ times if it was in fact January, but lying within the range .5 to 2.5 times as likely, suggesting

$$\mathbf{E}[\theta_1^*(0,m)] = .29 \qquad \text{for all } i \text{ and } m$$

$$Var[\theta_1^*(0,m)] = .09 \qquad \text{for all } i \text{ and } m$$

For $\mathbf{E}[\theta_0^*(0,m)]$ we shall use the form suggested in (4.5.3) with $\nu = 5$, $L = 1.1$, $U = 1.25$, $\eta_L = -0.5$, and $\eta_U = +0.5$. Giving

$$\mathbf{E}[\theta_0^*(0,m)] = \begin{cases} -0.4 & m \in (1.1, 1.13] \\ -0.2 & m \in (1.13, 1.16] \\ 0 & m \in (1.16, 1.19] \\ 0.2 & m \in (1.19, 1.22] \\ 0.4 & m \in (1.22, 1.25] \end{cases}$$

To set the variance we shall use the bounds indicated by the dotted lines in figure 4.5.5, yielding

$$Var[\theta_0^*(0,m)] = \begin{cases} 0.16 & m \in (1.1, 1.13] \\ 0.13 & m \in (1.13, 1.16] \\ 0.11 & m \in (1.16, 1.19] \\ 0.13 & m \in (1.19, 1.22] \\ 0.16 & m \in (1.22, 1.25] \end{cases}$$

Contracts will be arriving at fairly regular time intervals, so information will be discounted with time by using the constant discount matrix

$$B^* = \begin{pmatrix} .95 & 0 & 0 \\ 0 & .99 & 0 \\ 0 & 0 & .99 \end{pmatrix}^{-1}$$

Note that the majority of discounting of information is done via the basic setting θ_0, reflecting the fact that we believe that the prestige and seasonal effects are likely to be fairly stable over time.

If contracts arrive at non-regular time intervals it would be neccessary to make B^* a function of time t. This could be done by relating B^* to t in a linear way such as used for k_t in §4.2.

Table 4.5.6 lists the details of each of 60 contracts together with our chosen markup and the lowest markup from our competitors.

Figure 4.5.7 shows $\mathbf{E}[\theta_0(60, m)]$ ($= \mathbf{E}[\eta(60, m)]$) for company 1 together with the prior setting $\mathbf{E}[\eta(0, m)]$. The probability that we win the contract at time 60 (that is $\alpha_0'(60, m)/\alpha_*'(60, m)$) is shown in Figure 4.5.8. Figure 4.5.9 shows the corresponding expected percentage profit function ($prof(60, m)$) for the contract at time 60.

We see that initially the model tends to choose the same markup for several time periods. This is mainly due to the fact that in the early stages the model only has a small number of markups to choose from, and in part due to the continuing influence of the step function prior. However after about 30 contracts the model is starting to behave less predictably, and we see in figure 4.5.9 that at contract 60 the expected percentage profit function is starting to resemble a continuous curve.

We might conclude that with a vague step prior, such as that used in this example, the model should not be relied upon too heavily until at least 30 data points have been observed.

contract	prestige (π_i)	month (s_i)	chosen markup	m^*
1	0.5	1	1.190	1.179
2	0.5	1	1.190	1.163
3	0.8	2	1.190	1.141
4	0.5	3	1.190	1.188
5	0.2	4	1.188	1.204
6	0.2	5	1.188	1.196
7	0.8	5	1.188	1.153
8	0.5	6	1.188	1.162
9	0.5	7	1.179	1.189
10	0.3	8	1.179	1.185
11	0.9	8	1.179	1.139
12	0.7	9	1.179	1.145
13	0.5	10	1.179	1.194
14	0.5	11	1.179	1.176
15	0.6	11	1.176	1.149
16	0.2	12	1.176	1.183
17	0.1	2	1.176	1.209
18	0.7	2	1.179	1.155
19	0.9	3	1.176	1.144
20	0.5	4	1.130	1.181
21	0.6	4	1.139	1.166
22	0.5	5	1.139	1.190
23	0.5	6	1.139	1.154
24	0.4	7	1.139	1.165
25	0.4	7	1.139	1.187
26	0.9	8	1.130	1.125
27	0.6	9	1.139	1.162
28	0.5	10	1.139	1.177
29	0.1	10	1.144	1.201
30	0.5	11	1.139	1.143
31	0.2	12	1.139	1.208
32	0.6	1	1.162	1.140
33	0.9	2	1.141	1.110
34	0.3	3	1.140	1.198
35	0.5	3	1.139	1.187
36	0.8	4	1.143	1.183
37	0.1	5	1.139	1.213
38	0.5	6	1.143	1.200
39	0.6	6	1.143	1.159
40	0.5	7	1.144	1.179
41	0.4	8	1.144	1.158
42	0.3	9	1.143	1.158
43	0.9	9	1.153	1.116
44	0.5	10	1.144	1.173
45	0.2	11	1.143	1.214
46	0.6	12	1.153	1.170
47	0.7	12	1.153	1.151
48	0.5	1	1.153	1.148
49	0.5	1	1.144	1.152
50	0.4	2	1.148	1.162
51	0.3	3	1.148	1.219
52	0.7	4	1.153	1.132
53	0.5	5	1.151	1.198
54	0.6	5	1.152	1.158
55	0.5	6	1.153	1.141
56	0.8	6	1.148	1.216
57	0.5	7	1.152	1.196
58	0.1	7	1.151	1.196
59	0.4	8	1.151	1.192
60	0.5	9	1.153	1.219

TABLE 4.5.6

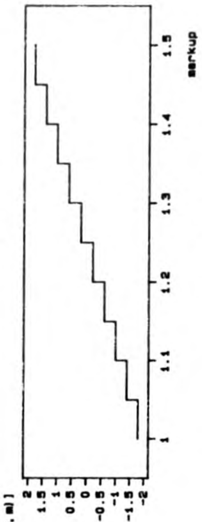


FIGURE 4.5.4

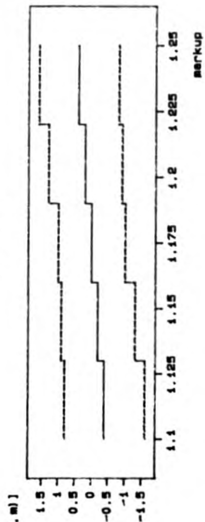


FIGURE 4.5.5

FIGURE 4.5.7

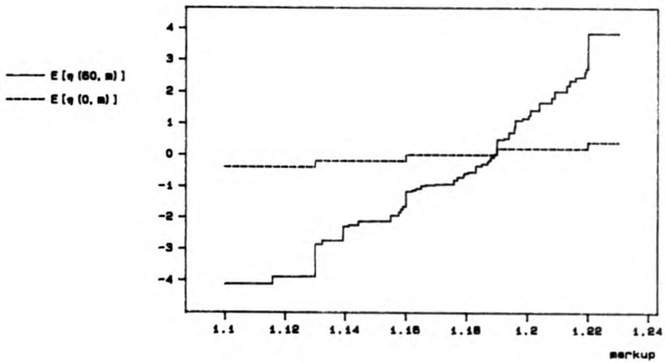
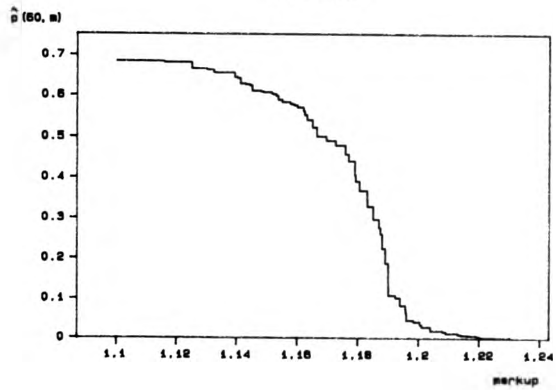
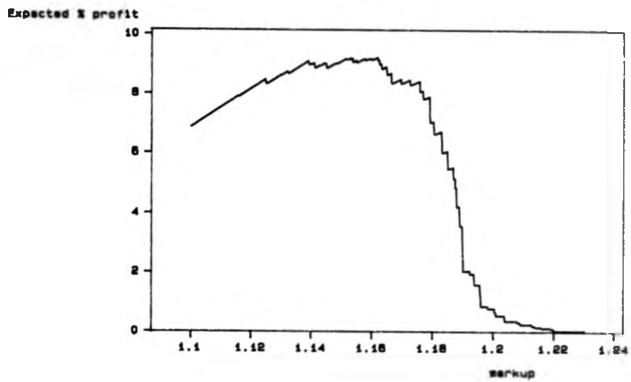


FIGURE 4.5.8



$\hat{p}$ (60, m)
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1.1
1.12
1.14
1.16
1.18
1.2
1.22
1.24
markup

FIGURE 4.5.9



5. A SEQUENTIAL FRIEDMAN TYPE MODEL

This chapter considers the same problem as chapter 4, that is once again we shall be assisting an individual bidder. We mirror the work of chapter 4 by presenting a sequential Friedman type model, that is, a model based directly on the random variables $X_1, \ldots, X_n$, the actual bids of our n competitors. We still make the same basic assumptions that $C \amalg p(m)$ and that the contract is awarded to the lowest bidder. A feature of the model is that it incorporates a dependency structure on our competitors bids by noting their common variation in cost estimates. Thus in this chapter we tend to work in terms of bids rather than markups.

The model requires information about who has been invited to bid on each contract as well as the actual value of bids made. Thus the model will be appropriate in an environment where we would expect to receive substantially more information about a contract than in the last chapter.

5.1 Bidding On a Single Contract.

One recurring feature of the litereature on the Friedman and Gates models is that the Gates formula performs better in simulations, see for example Benjamin & Meador (1979). This is not suprising since the Gates formula does at least impose some dependency on $X_1, \ldots, X_n$, although the form of this dependency is not easily identifiable. The Friedman formula assumes them to be independent, an assumption which is extremely unlikely to be true in practice. It thus seems sensible to attempt to isolate the source of dependency between $X_1, \ldots, X_n$, in order that we may put a more realistic prior on $\mathbf{X}$. Note that throughout this section we are assuming we have no specific knowledge of any one company, and thus are looking for a prior distribution to reflect only what is general knowledge or logically obvious about a competitors bids.

An easily identifiable type of dependency is a 'common environment' dependency similar to that discussed in the reliability litereature, see for example Lindley & Singpurwalla (1984). This form of dependency arises, since all our competitors bids will start with an assesment of the cost of the contract to them, see Ward & Chapman (1988). This cost asessment is in turn a function of external factors, such as raw material prices, which are applicable to all bidders.

Thus all the bids are functions of their 'common environment', in this case raw material prices. This suggests a competitors bid may be decomposed as follows;

$$X_i = CM_i \tag{5.1.1}$$

for $i = 1, \ldots, n$. Where C is a random variable representing the cost of the contract to the bidder, and is common to all bidders. M_i is a random variable representing the markup F_i puts on to his cost assesment for profit. Since it is assumed we are uninformed about the companies it is reasonable to take $M_1, \ldots, M_n$ to be exchangeable random variables.

Taking logs, (5.1.1) becomes;

$$\log X_i = \log C + \log M_i \tag{5.1.2}$$

The exchangeability of $M_1, \ldots, M_n$ means we can write $\log M_i = \log M' + \log M_i'$ where $M_1', \ldots, M_n'$ are independent identically distributed (i.i.d.) random variables. The lognormal is often a good distribution for cost random variables, so if it is assumed C is distributed lognormal, and that $\log M_i'$ and $\log M'$ are distributed locally normal, (5.1.2) is equivalent to the following normal random effects model for $\log X_i$, see Box & Tiao (1973).

$$\log X_i = \phi + \epsilon + \epsilon_i \tag{5.1.3}$$

where

$$\begin{aligned}
\phi &= \text{ constant unknown parameter } = \mathbf{E}[\log C + \log M_i] \\
\epsilon &= (\log M'C - \mathbf{E}[\log M'C]) \sim N(0, \sigma^2), \text{ say} \\
\epsilon_i &= (\log M_i' - \mathbf{E}[\log M_i']) \sim N(0, \tau^2), \text{ say}
\end{aligned}$$

and $\epsilon_1, \ldots, \epsilon_n$ are i.i.d. random variables. Which is equivalent to having

$$(\log X_1, \ldots, \log X_n) \sim N_n(\phi \mathbf{e}_n, \Sigma) \tag{5.1.4}$$

where $\mathbf{e}_n$ is the $n \times 1$ vector of ones, and Σ is given by

$$\Sigma_{n \times n} = \tau^2 I_n + \sigma^2 \mathbf{e}_n \mathbf{e}_n^T$$

a so called 'intraclass' matrix, I_n being the $n \times n$ identity matrix. This is in turn the same as saying $\log X_1, \ldots, \log X_n$ are exchangeable random variables.

It is worth noting that many authors have assumed normality for costs and bids and not lognormality as in (5.1.4), see for example Naert & Weverbergh (1978). There is however rarely, if ever, any attempt to justify this assumption.

It will prove convenient to consider the following reparameterisation. Let ρ and v be defined by

$$\rho = \frac{\sigma^2}{\sigma^2 + \tau^2} \quad \text{and} \quad v = \sigma^2 + \tau^2$$

then clearly

$$\begin{aligned} \rho &= \text{ correlation } (\log X_i, \log X_j) > 0 && \text{for all } i,j \\ \text{and} \qquad v &= \text{ variance } (\log X_i) && \text{for all } i \end{aligned}$$

Consider now computing the probability that we win the contract with bid b when competing against $F_1, \ldots, F_n$, i.e. $\bar{p}(b)$. Where

$$\begin{aligned} \bar{p}(b) &= p(b < X_1 \cap \ldots \cap b < X_n) \\ &= p(\log b < \log X_1 \cap \ldots \cap \log b < \log X_n) \end{aligned}$$

Of course $\bar{p}(b) = \bar{p}(b, \rho, v) = \bar{p}(b, \sigma^2, \tau^2)$. The parameter we are particularly interested in is ρ, since this allows us to quantify the dependency between competing companies, thus we will denote $\bar{p}(b)$ as $\bar{p}(b, \rho)$, suppressing the dependency on v.

Firstly we note that $\bar{p}(b, \rho)$ is simply an orthant probability for the multivariate normal random variable $\log X_1, \ldots, \log X_n$. So it can be computed using any one of the standard numerical techniques for calculating such probabilities, see for example the methods outlined in Johnson & Kotz (1972). However we can also use the following form which will prove useful later,

$$\begin{aligned} \hat{p}(b, \rho) &= \int_{-\infty}^{\infty} \left[\Phi(z \sqrt{\frac{\rho}{1-\rho}} + \frac{\phi - \log b}{\sqrt{v(1-\rho)}}) \right]^n d\Phi(z) \\ &\equiv \mathbf{E} \left[\{ \Phi(Z \sqrt{\frac{\rho}{1-\rho}} + \frac{\phi - \log b}{\sqrt{v(1-\rho)}}) \}^n \right] \end{aligned} \tag{5.1.5}$$

where $Z \sim N(0,1)$ and Φ is the standard normal distribution function. This is easily shown as follows.

$$
\begin{aligned}
\hat{p}(b,\rho) &= p(b < min(X_1, \dots X_n)) \\
&= p(\log b < \phi + \epsilon + min(\epsilon_1, \dots, \epsilon_n)) \qquad \text{from (5.1.3)}, \\
&= \int_{\epsilon=-\infty}^{\infty} p(min(\epsilon_1, \dots, \epsilon_n) > \log b - \phi - \epsilon) d\Phi(\epsilon/\sigma) \qquad \text{since } \epsilon \sim N(0, \sigma^2), \\
&= \int_{-\infty}^{\infty} \left[1 - \Phi(\frac{\log b - \phi - \epsilon}{\tau})\right]^n d\Phi(\epsilon/\sigma)
\end{aligned}
$$

since $\epsilon_1, \dots, \epsilon_n$ are i.i.d. $N(0, \tau^2)$. The result follows by putting $z = \epsilon/\sigma$ and reparameterising in terms of ρ and v.

We can now see that an alternative way of computing $\hat{p}(b,\rho)$ is to use a Monte-Carlo method, namely, generate a random sample $z_1, \dots, z_n$ from a $N(0,1)$ distribution and then an unbiased consistent estimator of $\hat{p}(b,\rho)$ is $\overline{u} = \sum_{i=1}^n u_i/n$, where

$$
u_i = \left[\Phi(\, z_i \sqrt{\frac{\rho}{1-\rho}} + \frac{\phi - \log b}{\sqrt{v(1-\rho)}}\,)\right]^n
$$

We now briefly consider a generalisation of the above model. It may be that we can identify different companies as coming from different 'environments'. For example we may know that there are only two possible suppliers of raw materials. Thus all bidders using the first supplier will obey (5.1.1) with cost random variable C_1 say, and all bidders using the second supplier will obey (5.1.1) with cost random variable C_2. In general we might have J environments, whence (5.1.1) becomes

$$
X_{ij} = C_j M_{ij}
$$

where X_{ij} is a random variable representing the bid of the i^{th} member of the j^{th} environment, M_{ij} is his markup and C_j is the cost random variable for the j^{th} environment, $i = 1, \dots, r_j$, $j = 1, \dots, J$. Clearly (5.1.2) now becomes

$$
\log X_{ij} = \phi_j + \epsilon_j + \epsilon_{ij} \tag{5.1.6}
$$

where for all i, j

$$
\begin{aligned}
\phi_j &= \text{ constant unknown parameter } = \mathbf{E}[\log C_j + \log M_{ij}] \\
\epsilon_j &= (\log M_j' C_j - \mathbf{E}[\log M_j' C_j]) \sim N(0, \sigma_j^2) \\
\epsilon_{ij} &= (\log M_{ij}' - \mathbf{E}[\log M_{ij}']) \sim N(0, \tau_j^2)
\end{aligned}
$$

This may now be described as a normal 'exchangeable blocks' model for log bids, since, as before, it is equivalent to having $\log X_{ij} \sim N(\phi_j, v_j)$ for all i, j with $\log X_{1j}, \ldots, \log X_{r_j j}$ being exchangeable for all j. That is, log bids are exchangeable only within their own block, or environment.

If we go on to make the assumption that bids in different blocks are independent then $\hat{p}(b, \rho)$ is simply given by

$$\hat{p}(b, \rho) = \prod_{j=1}^{J} \hat{p}^j(b, \rho)$$

where $\hat{p}^j(b, \rho)$ is the probability of beating all competitors in block j, and is computed as before. Hence we need only analyse further the one block model.

Before moving on to the sequential model we briefly consider a comparison between the above exchangeable model and the Friedman and Gates models with a lognormal distribution on bids.

Define the probability of winning the contract with bid b to be $\hat{p}_F(b), \hat{p}_G(b)$ and $\hat{p}(b, \rho)$ when using the Friedman, Gates and exchangeable models respectively.

In the case considered previously, where we are equally uninformed about all of our competitors, we have

$$\hat{p}_F(b) = q^n \qquad \hat{p}_G(b) = \left[1 + \frac{n(1-q)}{q}\right]^{-1}$$

and $\hat{p}(b, \rho)$ is computed using (5.1.5). Where, from (5.1.4)

$$q = p(b < X_i) = \Phi\left(\frac{\phi - \log b}{\sqrt{v}}\right) \qquad \text{for all } i$$

The following lemma provides some insight in to the relative behaviour of $\hat{p}_F(b), \hat{p}_G(b)$ and $\hat{p}(b, \rho)$

Lemma 5.1.7.

For any given ϕ, v and $n > 1$, we have;

(1) *$\hat{p}(b, \rho)$ is an increasing function of ρ, for all b*

(2) *$\hat{p}_F(b) = \hat{p}(b, 0)$ for all b*

(3) *$\hat{p}_G(e^\phi) = \hat{p}(e^\phi, \frac{1}{2}) = \frac{1}{n+1}$*

(4) *$\hat{p}(b, 0) < \hat{p}_G(b) < \hat{p}(b, 1)$ for all b, $|b| < \infty$*

PROOF:

(1) The result follows directly from Slepians inequality, see Tong (1980).

(2) from (5.1.5) $\bar{p}(b,0) = \mathbf{E}[\{\Phi(\frac{\phi - \log b}{\sqrt{v}})\}^n] = q^n = \bar{p}_F(b)$.

(3) again from (5.1.5) $\bar{p}(e^\phi, \frac{1}{2}) = \mathbf{E}[(\Phi(Z))^n] = \frac{1}{n+1}$, since $\Phi(Z)$ is ditributed uniform on $[0,1]$. Also $\hat{p}_G(e^\phi) = \frac{1}{n+1}$ since $q = \frac{1}{2}$ when $b = e^\phi$.

(4) from (5.1.3) we have $\bar{p}(b,1) = q$ since $\epsilon_i = 0$ for all i when $\rho = 1$. So

$$\hat{p}_G(b) = \left[1 + \frac{n(1-q)}{q}\right]^{-1} = \frac{q}{q.1 + (1-q).n} < q = \bar{p}(b,1) \text{ for } n > 1$$

also

$$q^{-n} \equiv \left[1 + \frac{1-q}{q}\right]^n = 1 + \frac{n(1-q)}{q} + \text{ positive terms } > 1 + \frac{n(1-q)}{q} \text{ for } n > 1$$

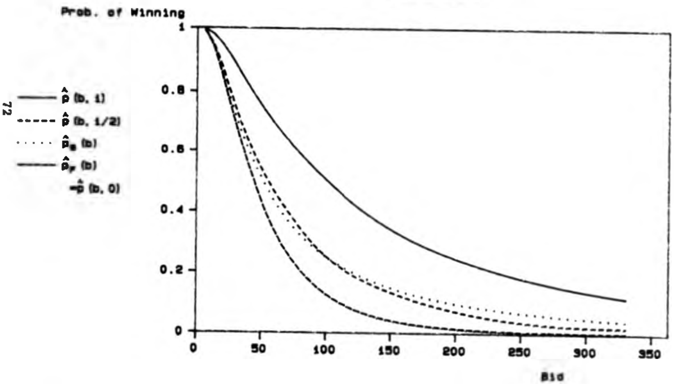
thus

$$\bar{p}(b,0) = q^n < \left[1 + \frac{n(1-q)}{q}\right]^{-1} = \hat{p}_G(b)$$ □

Figure 5.1.8 shows $\bar{p}_F(b)$, $\hat{p}_G(b)$, $\bar{p}(b, \frac{1}{2})$ and $\bar{p}(b,1)$ for the specific case $\phi = \log 100, v = 1$ and $n = 3$. However lemma (5.1.7) tells us that this will be the form of these functions for all ϕ, v and n, with $\bar{p}(b,\rho)$ for $\rho \neq \frac{1}{2}$ being similar to $\bar{p}(b, \frac{1}{2})$, but shifted either up or down depending on whether $\rho > \frac{1}{2}$ or $\rho < \frac{1}{2}$ respectively.

Bearing in mind that we are likely to be bidding near to, or slightly below e^ϕ, we conclude that when we are equally uninformed about our competitors, use of the Gates model is similar to using the exchangeable model with correlation of $\frac{1}{2}$ or slightly below $\frac{1}{2}$. Whilst use of the Friedman formula is identical to using the exchangeable model with zero correlation.

FIGURE 5.1.8



Prob. of Winning
1
0.8
0.6
0.4
0.2
0
0
50
100
150
200
250
300
350
Bid
$\hat{p}$ (b, 1)
$\hat{p}$ (b, 1/2)
$\hat{p}_s$ (b)
$\hat{p}_F$ (b)
$\hat{p}$ (b, 0)

5.2 Bidding on a Sequence of Contracts.

We consider first the case where our company is bidding on a sequence of identical contracts i.e. the cost distribution is the same at each time period. It is then shown how the results obtained for this case are easily extended to a sequence of contracts with different costs.

Ideally we would like to use the model derived for $\mathbf{X}$ in the previous section purely as a prior setting, and then allow ourselves to learn about each X_i completly freely. That is, let each X_i have a different mean and variance, and each X_i, X_j pair have a different correlation. An appropriate analysis may be the standard Normal-Wishart analysis, see De Groot(1970). The problem with this type of model lies in the form of data we are to receive. If we have n competitors, then at each time period we will observe the bids of a subset of these competitors, and of course a different subset at each time period. We will describe this as 'censored' data, since at each time period we can view our observations as a censored value of the single multivariate observation $X_1, \ldots, X_n$. It is quickly seen that the structure of a free running model, such as the Normal-Wishart, breaks down with censored data, and leads to calculations which are extremely difficult even numerically. Note that this censoring is of course not random, since it is determined by the tendering organisations choice of companies to bid. There is, thus, the possibility of learning about the tendering organisations policy in selecting companies to bid. A model that can cope with this censored data is also of course capable of coping with missing observations. This may well prove useful, as the possibility of not being able to find out a companies bid is quite real.

For the rest of this paper we make the assumption that the exchangeable blocks structure remains over time. This assumption will enable us to cope with the censored data mentioned above. If each competitor is in a separate block this model will be equivalent to the Friedman model with the lognormal distribution on bids, since competitors bids will be independent. As in the previous section we need only analyse one block.

Let $X_i(t)$ be the bid of the i^{th} competitor, in the particular block being considered, at time t, $i = 1, \ldots, n$, and let $Y_i(t) = \log X_i(t)$. Then our likelihood at time t is specified by

$$Y_1(t), \ldots, Y_{r_t}(t) \sim N_{r_t}(\phi(t)\mathbf{e}_{r_t}, \Sigma(t)) \qquad (5.2.1)$$

where $\Sigma(t) = \sigma^2(t) I_{r_t} + \tau^2(t)\mathbf{e}_{r_t}\mathbf{e}_{r_t}^T$ is the intraclass matrix specified previously, and r_t is the number of bids we are observing from this block at time t. This suggests that for our Bayesian analysis we will need a prior distribution across $\phi(0), \tau^2(0)$ and $\sigma^2(0)$. However, an important simplification to our likelihood can be made. As noted previously, at $t = 0$, (5.2.1) is equivalent to the model

$$Y_i(0) = \phi(0) + \epsilon(0) + \epsilon_i(0) \tag{5.2.2}$$

where $\phi(0)$ is a 'fixed effect' and $\epsilon(0) \sim N(0, \sigma^2(0))$ is a 'random effect'. Since we are about to put a prior on $\phi(0)$ and thus effectively make it also a random effect, $\epsilon(0)$ can be interpreted as a fixed effect with prespecified prior distribution $\epsilon(0) \sim N(0, \sigma^2(0))$. So (5.2.2) can be rewritten

$$Y_i(0) = \phi'(0) + \epsilon_i(0)$$

where $\phi' = \phi + \epsilon$ is a fixed effect. This leads to the simplified likelihood,

$$Y_1(t), \ldots, Y_{r_t}(t) \sim N_{r_t}(\phi'(t)\mathbf{e}_{r_t}, \tau^2(t) I_{r_t}) \tag{5.2.3}$$

$\sigma^2(0)$ is no longer relevant, as it does not appear in the likelihood, however we are compelled to put a normal prior on $\epsilon(0)$, so consider the following prior structure;

$$\phi'(0)|\tau^2(0) \sim N(m_0, \frac{\tau^2(0)}{\alpha_0}) \qquad \tau^2(0) \sim G^{-1}(a_0, b_0) \tag{5.2.4}$$

where m_0, α_0, a_0, b_0 are prior parameters, and G^{-1} denotes the inverse Gamma distribution with p.d.f., $p(\tau^2) \propto \tau^{-2(1+a)} exp(-b/\tau^2)$, which is a commonly used prior for variances. Note that this prior, which is in fact conjugate for the likelihood (5.2.3), also includes the standard uninformative priors for ϕ' and τ^2, for when $\alpha_0 = a_0 = b_0 = 0$ (5.2.4) becomes

$$p(\phi'(0)) \propto 1 \qquad p(\tau^2(0)) \propto \frac{1}{\tau^2(0)}$$

Define (5.2.4) as being equivalent to saying $\phi'(0), \tau^2(0) \sim NG^{-1}(m_0, \alpha_0, a_0, b_0)$. If we now make the inductive assumption

$$\phi'(t-1), \tau^2(t-1)|\mathbf{y}^{t-1} \sim NG^{-1}(m_{t-1}, \alpha_{t-1}, a_{t-1}, b_{t-1})$$

where $\mathbf{y}^{t-1}$ is all the data up to time $t-1$, Bayes theorem tells us that

$$\begin{aligned} p(\phi'(t), \tau^2(t)|\mathbf{y}(t), \mathbf{y}^{t-1}) &\propto p(\mathbf{Y}(t)|\phi'(t), \tau^2(t)) \cdot p(\phi'(t), \tau^2(t)|\mathbf{y}^{t-1}) \\ &\propto p(\mathbf{Y}(t)|\phi'(t), \tau^2(t)) \cdot p(\phi'(t-1), \tau^2(t-1)|\mathbf{y}^{t-1}) \end{aligned}$$

which, with a little manipulation, yields

$$\phi'(t), r^2(t)|\mathbf{y}(t), \mathbf{y}^{t-1} \sim NG^{-1}(m_t, \alpha_t, a_t, b_t)$$

where

$$a_t = a_{t-1} + \frac{r_t}{2} \qquad \alpha_t = \alpha_{t-1} + r_t \qquad m_t = \frac{m_{t-1}\alpha_{t-1} + r_t \bar{y}(t)}{\alpha_{t-1} + r_t}$$

$$b_t = b_{t-1} + \frac{1}{2}(\alpha_{t-1}m_{t-1}^2 - \alpha_t m_t^2 + \mathbf{y}^T(t)\mathbf{y}(t)) \tag{5.2.5}$$

The crux of our problem is, given the observation at the current time period, to compute the probability we beat a given number of competitors, r_{t+1} say, from this block at the *next* time period. This necesitates calculating a predictive distribution i.e. the distribution of $\mathbf{Y}(t+1)|\mathbf{y}^t$. This is given by

$$p(\mathbf{Y}(t+1)|\mathbf{y}^t) = \int_0^\infty \int_{-\infty}^\infty p(\mathbf{Y}(t+1)|\phi', r^2)p(\phi', r^2|\mathbf{y}^t)d\phi' dr^2$$

which yields

$$\mathbf{Y}(t+1)|\mathbf{y}^t \sim T_{2a_t}(m_t \mathbf{e}_{r_{t+1}}, S_t) \tag{5.2.6}$$

a multivariate t-distribution with $2a_t$ degrees of freedom and

$$\mathbf{E}[\mathbf{Y}(t+1)|\mathbf{y}^t] = m_t \mathbf{e}_{r_{t+1}} \qquad Cov[\mathbf{Y}(t+1)|\mathbf{y}^t] = S_t = \frac{b_t}{a_t - 1}\left(I_{r_{t+1}} + \frac{\mathbf{e}_{r_{t+1}}\mathbf{e}_{r_{t+1}}^T}{\alpha_t}\right)$$

Before discussing the problem of calculating the probability of beating the r_{t+1} competitors in this block, we look at a couple of generalisations of the predictive distribution (5.2.6).

Firstly we return to the problem of bidding on a sequence of contracts with different cost random variables. Our model already assumes that a company's markup is independent of the cost of the contract. If we go on to assume that the $Var[\log C_t]$ is constant, then we can standardise with respect to the mean and variance of the log cost distribution. Thus our model becomes

$$Y_i^s(t) = \phi''(t) + \epsilon_i'(t)$$

where

$$Y_i^s(t) = \text{ standardised data } = \frac{\log X_i(t) - l_t}{g}$$

$$\phi''(t) = \frac{\phi'(t) - l_t}{g} \qquad \epsilon_i'(t) = \frac{\epsilon_i(t)}{g}$$

$l_t = \mathbf{E}[\log C_t]$ and $g^2 = \text{Var}\,[\log C_t]$. Our updating of parameters is exactly as in (5.2.5) with the standardised data replacing $\mathbf{y}(t)$. Not forgetting that the parameters m_t, α_t, a_t, b_t now relate to the random variables $\mathbf{Y}^*(t)$ and $\phi''(t)$. The predictive distribution (5.2.6) now becomes

$$\mathbf{Y}(t+1)|\mathbf{y}^t \sim T_{2a_t}((l_{t+1} + g m_t)\mathbf{e}_{r_{t+1}}, g^2 S_t) \tag{5.2.7}$$

Recalling that C_t is distributed lognormal and so $\log C_t$ is distributed normal, if $c_t = \mathbf{E}[C_t]$, then we have $l_t = \log c_t - g^2/2$ Also, $Var[C_t] = c_t^2(e^{g^2} - 1)$, thus our assumption about constant log cost variance is equivalent to saying $Var[C_t]$ is directly proportional to c_t^2. A more precise interpretation of the parameters m_t, α_t, a_t, b_t, and suggested prior settings, are given in the next section.

Exactly as in §4.2 it is desirable to model a companies behaviour changing with time. A natural way to model this drift is to let the parameters $\phi''(t), \tau^2(t)$ develop with the power steady model with parameter k, this yields

$$\phi''(t), \tau^2(t)|\mathbf{y}^{t-1} \sim NG^{-1}(m_{t-1}, k\alpha_{t-1}, ka_{t-1} + \frac{3}{2}(k-1), kb_{t-1})$$

the updating in (5.2.5) then becomes

$$a_t = ka_{t-1} + \frac{3}{2}(k-1) + \frac{r_t}{2} \qquad \alpha_t = k\alpha_{t-1} + r_t \qquad m_t = \frac{m_{t-1}k\alpha_{t-1} + r_t\bar{y}(t)}{k\alpha_{t-1} + r_t}$$

$$b_t = kb_{t-1} + \frac{1}{2}(k\alpha_{t-1}m_{t-1}^2 - \alpha_t m_t^2 + \mathbf{y}^T(t)\mathbf{y}(t)) \tag{5.2.8}$$

and the predictive distribution becomes

$$\mathbf{Y}(t+1)|\mathbf{y}^t \sim T_{2ka_t+3(k-1)}((l_{t+1} + g m_t)\mathbf{e}_{r_{t+1}}, g^2 S_t') \tag{5.2.9}$$

$$S_t' = \frac{2kb_t}{2ka_t + 3k - 5}\left(I_{r_{t+1}} + \frac{\mathbf{e}_{r_{t+1}}\mathbf{e}_{r_{t+1}}^T}{k\alpha_t}\right)$$

This is the most general form of predictive distribution we shall look at.

Consider now the setting of the parameter k. First we note that values of k close to 1 represent a small amount of drift of information, and it is in this region that we are likely to choose k. Since the drift of information is steady the parameter k should clearly be related to the time between contracts. Consider for example setting

$$1 - k = (1 - \lambda)\frac{\delta}{\delta_0}$$

where δ is the time interval between the tendering of the last contract and the tendering of this one, and δ_0 is some base time period, for example the average time between contracts. Note, if two contracts are put out simultaenously, then $\delta = 0$, $k = 1$ and there is no discounting of information. The parameter λ thus represents the drift in information over the constant time interval δ_0. If δ_0 is reasonably small, values of λ as high as .99 may not be inappropriate.

We now consider the problem of computing the probability that we beat r_{t+1} competitors in a given block, with bid b, denote this by $\hat{p}_t(b)$ so

$$\hat{p}_t(b) = p(\log b < Y_1(t+1)\cap, \ldots, \cap \log b < Y_{r_{t+1}}(t+1)|\mathbf{y}^t)$$

where $\mathbf{Y}(t+1)|\mathbf{y}^t$ is distributed as in (5.2.9). The following provides a method for doing this.

$$\begin{aligned}\hat{p}_t(b) &= \int_{-\infty}^{\infty}\int_0^{\infty}\left[\Phi(z\sqrt{\frac{k\alpha_t}{k\alpha_t+1}} - \frac{(\log b - \mu_t)\sqrt{s}}{\sqrt{\nu_t d_t}}\right]^{r_{t+1}} dF_S(s)d\Phi(z) \\ &= \mathbf{E}_{Z,S}\left[\{\Phi(Z\sqrt{\frac{k\alpha_t}{k\alpha_t+1}} - \frac{(\log b - \mu_t)\sqrt{S}}{\sqrt{\nu_t d_t}})\}^{r_{t+1}}\right] \qquad (5.2.10)\end{aligned}$$

where

$$\nu_t = 2k\alpha_t + 3(k-1), \quad \mu_t = \mathbf{E}[Y_i(t+1)|\mathbf{y}^t] = l_{t+1} + gm_t$$

$$d_t = \frac{\nu_t - 2}{\nu_t}Var(Y_i(t+1)|\mathbf{y}^t) = \frac{2b_t(k\alpha_t+1)g^2}{\alpha_t(2k\alpha_t + 3(k-1))}$$

and $Z \sim N(0,1)$, $S \sim \chi^2_{\nu_t}$.

This is shown by noting that $Cov(\mathbf{Y}(t+1))$ has an intraclass structure. This means, from the definition of the multivariate t-distribution, we can write

$$Y_i(t+1) - \mu_t = \frac{(\eta(t) + \eta_i(t))\sqrt{\nu_t}}{\sqrt{\xi(t)}}$$

where $\eta(t) \sim N(0, d_t k\alpha_t/(k\alpha_t+1))$, $\eta_i(t) \sim N(0, d_t)$, $\xi(t) \sim \chi^2_{\nu_t}$, and $\eta, \eta_1, \ldots, \eta_n, \xi$ are mutually independent. The result follows by similar manipulations to those used in establishing (5.1.5).

Thus, as in §5.1, we can use a Monte-Carlo method to compute $\hat{p}_t(b)$. If we generate an independent random sample $(z_1, s_1), \ldots, (z_n, s_n)$ from a $N(0,1)$ and $\chi^2_{\nu_t}$ distribution, then an estimate of $\hat{p}_t(b)$ is $\bar{u} = \sum u_i/n$ where

$$u_i = \left[\Phi(z_i\sqrt{\frac{k\alpha_t}{k\alpha_t+1}} - \frac{(\log b - \mu_t)\sqrt{s_i}}{\sqrt{\nu_t d_t}})\right]^{r_{t+1}}$$

Note that to generate a $\chi^2_{\nu_t}$ observation we use the result that if $Z_1, \ldots, Z_m$ are i.i.d $N(0,1)$ then $\sum_1^m Z_i^2 \sim \chi^2_m$.

If $r_{t+1} = 1$ our predictive distribution is of course a univariate t-distribution, and so the Monte-Carlo method is unneccesary. Indeed, since ν_t is likely to be large, see the next section, the following approximation is very good;

$$\hat{p}_t(b) \simeq 1 - \Phi\left(\zeta_t \sqrt{\frac{2\nu_t - 1}{3}} \sinh^{-1} \sqrt{\frac{3(\log b - \mu_t)^2}{2\nu_t d_t}} \right) \tag{5.2.11}$$

where ζ_t=sign of $(\log b - \mu_t)$.

It is now neccessary to compute our optimal bid for time $t+1$ by maximising the expected profit function $prof(b) = (b - c_{t+1})\hat{p}_t(b)$. The suggested method for doing this is to compute $\hat{p}_t(b)$ at a small number of points, 6 or 8 maybe, using the Monte-Carlo method outlined above, and then fit cubic splines through these points, see Atkinson (1978) pp 143–149 for a simple account of how to do this. Once these splines have been fitted, any number of values of $\hat{p}_t(b)$ are easily computed. The maximum is then obtained using some simple direct search method on $prof(b)$, or even graphically.

5.3 An Illustrative Example.

At every time period we are to compete against 4 companies whom we have decided will come from 2 exchangeable blocks. Before setting the prior parameters m_0, α_0, a_0, b_0, for each block, we set $g^2 = Var[\log C_t]$. The way to do this is simply to state an interval, (L,U), in which we are almost certain $\log C_t$ will lie. An estimate of g is then given by $g = (U - L)/6$, using the fact that $\log C_t$ is normal, and normal random variables are almost certain to lie within 3 standard deviations of their mean. Looking at our cost estimates below we shall thus use $g = 0.2$. For the purposes of this example this is a somewhat rough and ready estimate, a practitioner would presumably be able to specify a reasonably accurate range for $\log C_t$. As noted earlier l_t is now given by $l_t = \log c_t - g^2/2$.

To set m_0, α_0, a_0, b_0, we note that for the general predictive distribution (5.2.9) these parameters have the following interpretations. First, $m_0 = \mathbf{E}[\log M_t]/g$, where M_t is the markup taken by a company in this block. For these fairly vague prior settings it is not unreasonable to take $\mathbf{E}[\log M_t] \simeq \log \mathbf{E}[M_t]$, and so, taking $\mathbf{E}[M_t] = 1.15$, we shall set $m_0 = 0.7$ for both blocks. Large values of α_0 correspond to greater confidence in our setting of m_0, since the prior on ϕ'' becomes more peaked. Indeed we have already mentioned that $\alpha_0 = 0$ corresponds to an uninformative, uniform prior on ϕ''. We shall take the relatively uninformative setting $\alpha_0 = 1$. To set a_0, b_0 we note that they satisfy the following equations;

$$\frac{b_0}{a_0 - 1} = \mathbf{E}[\tau^2] \qquad \frac{b_0^2}{(a_0 - 1)^2(a_0 - 2)} = Var[\tau^2]$$

where $\tau^2 = Var[\log \text{bid}]$. So we will want $\mathbf{E}[\tau^2]$ to be slightly larger than g^2, since we expect the variance of bids to be greater than the variance of costs, and $Var[\tau^2]$ to be smaller than g^2. This suggests a_0 should be reasonably large, of course very large values reflect greater confidence in prior settings. We shall take $b_0 = 1$ and $a_0 = 15$. Finally we shall assume contracts arrive at regular time intervals, and that there is a small steady development, which we shall model by taking $k = .99$.

Details of the costs of the seven contracts we will consider are given in the following table. All costs and bids are in £100's.

t	1	2	3	4	5	6	7
c_t	100	110	105	90	120	80	90
l_t	4.585	4.680	4.634	4.480	4.767	4.362	4.480

A summary of the analysis showing the updating of the parameters and the observed bids for each block are shown in tables 5.3.1 and 5.3.2. Details of our optimal bid at each time period are shown in the table below.

t	1	2	3	4	5	6	7
Optimal Bid (b)	106.3	118.4	113.7	97.5	129.7	86.4	97.3
$\hat{p}(b)$	.34	.47	.54	.57	.51	.57	.55
Expected profit	2.1	3.9	4.7	4.3	4.9	3.7	4.0

To illustrate the model in action we look in detail at the computation of the optimal bid for time $t = 3$. If $p^i(b) = p$(we beat all competitors in block i with bid b), $i = 1, 2$, then using (5.2.11) for block 1 and the Monte-Carlo method for block 2 the following probabilities were computed.

Bid	105	110	115	117.5	120	122.5	125	130
$p^1(b)$	.992	.950	.804	.680	.533	.384	.254	.086
$\hat{p}^2(b)$	.925	.795	.570	.440	.317	.215	.135	.045

Figure 5.3.3 shows the smoothed curves obtained by fitting cubic splines through these points. Note that $\bar{p}(b) = p($ we beat *all* competitors with bid $b) = p^1(b)p^2(b)$.

Figure 5.3.4 shows the expected profit function, $(b - 105)\bar{p}(b)$, for which the maximum is achieved at $b = 113.7$.

We see that our prior settings lead us to start off bidding quite conservatively. The result being that we win all the early contracts, with relatively small expected profits. However as we learn about our competitors we start to bid more adventurously and increase our expected profit, at least as a percentage of cost.

At first sight these results may seem strange, since a model which instructs us to bid so consistently low initially will, in practice, force our competitors to reduce their bids resulting in lower profits all round. However the important point is that after a number of contracts we are bidding more competitively, suggesting that the initial bids are due to the reasonably vague settings of m_0, α_0, a_0, b_0. In a practical situation we would expect to have past data on contracts. This could be used to arrive at more sensible settings of m_0, α_0, a_0, b_0 simply by starting with vague settings and running the past data through the model – once the model has settled down and is producing sensible bids the corresponding values of m_t, α_t, a_t, b_t can be used as the prior setting.

BLOCK 1								
t	0	1	2	3	4	5	6	7
r_t		2	2	1	1	2	1	1
Observed bids		113 118	124 130	120	102	137 139	89	101
Standardised bids		.711 .928	.699 .935	.768	.726	.762 .835	.633	.677
m_t	.7	.780	.795	.790	.781	.785	.769	.761
α_t	1	2.99	4.96	5.91	6.85	8.78	9.69	10.60
a_t	15	15.84	16.66	16.98	17.30	18.11	18.41	18.71
b_t	1	1.006	1.011	1.001	.993	.985	.985	.979

TABLE 5.3.1

BLOCK 2								
t	0	1	2	3	4	5	6	7
r_t		2	2	3	3	2	3	3
Observed bids		111 123	119 135	115 121 130	103 104 115	134 152	88 92 93	95 104 110
Standardised bids		.622 1.135	.493 1.124	.555 .809 1.168	.775 .823 1.326	.652 1.282	.577 .799 .853	.370 .823 1.103
m_t	.7	.819	.815	.826	.867	.883	.856	.841
α_t	1	2.99	4.96	7.91	10.83	12.92	15.60	18.44
a_t	15	15.84	16.66	17.98	19.29	20.08	21.36	22.63
b_t	1	1.066	1.155	1.239	1.344	1.438	1.469	1.601

TABLE 5.3.2

FIGURE 5.3.3

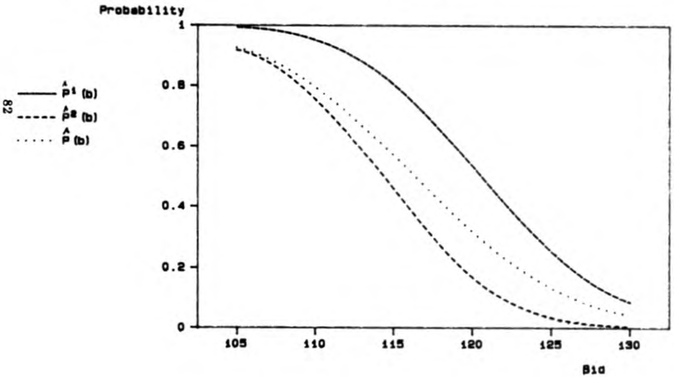
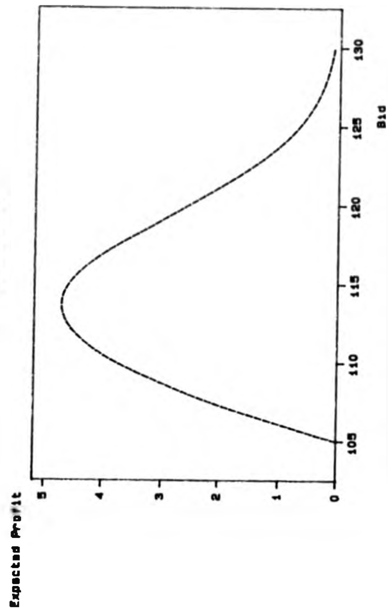

Probability
1
0.8
0.6
0.4
0.2
0
105
110
115
120
125
130
Bid
$\hat{p}^1$ (b)
$\hat{p}^2$ (b)
$\hat{p}$ (b)

FIGURE 5.3.4



6. RECONCILING THE FRIEDMAN AND GATES MODELS

In this chapter we consider ourselves to be an external observer attempting to model the behaviour of all the competing bidders. The main question we answer is motivated by the fact the Dirichlet Gates type model, now parameterised by $\alpha_0, \ldots, \alpha_n$ which are no longer functions of m, can be fully updated with just the knowledge of the identity of the winner of each contract. The question is, what Friedman type model would also require only this information to fully update itself ? In the next section we show that this is not possible with any non–degenerate countably additive distribution across $X_1, \ldots, X_n$. In §6.2 we show that it is possible with a finitely additive distribution. Properties of these distributions are then explored.

6.1 An Impossibility Theorem.

Consider the Dirichlet Gates type model of §4.1 used as a model for an external observer. In this case the model can work on the formal hypothesis that for each contract tendered we are only told the identity of the winner. In practice a company will often have more information than this. So we might ask the question as to when we are formally justified in discarding all information other than $r(t)$, and thus justified in using the model of §4.1 even when we have extra information ? For this we need to define a general class of models. An obvious choice is the class of Friedman type models described by a distribution across the actual bids of n companies $X_1, \ldots, X_n$.

Define the vector $\mathbf{I} = I_1, \ldots, I_n$ to rank the order of bids $X_1, \ldots, X_n$. Thus $I_i = j$ implies that company F_j produced the i^{th} lowest bid. A slight generalisation of our question is now whether, for a single contract, there are probability distributions across the n–tuple of company's bids $\mathbf{X} = X_1, \ldots, X_n$ such that $\boldsymbol{\theta}$, the vector of probabilities of who wins this contract, is independent of X_i, $1 \leq i \leq n$, given an arbitary function of $\mathbf{I}$? If this can be answered in the affirmitive then the types of models used in chapter 4 will also be formally justifiable models to use when we have additional information available to us. An obvious rider to this question is — what are the characteristics of such distributions, and do they provide realistic looking models that at least approximate structures we might expect to see in the bidding environment?

First suppose the whole of $\mathbf{I}$ is known to all competitors. Because the interpretation we have chosen for the vector $\boldsymbol{\theta}$ only concerns statements about the order statistics $\mathbf{Y} = (Y_1, \ldots, Y_n)$ of $\mathbf{X}$, where $Y_i = X_{(i)}$, $1 \leq i \leq n$, it is convenient, as in Lane & Sudderth (1979), to construct the distribution of $\mathbf{X}$ from a distribution on the pair $(\mathbf{I}, \mathbf{Y})$ where $\mathbf{I}$ indicates the order of the components of $\mathbf{X}$ and is defined above.

Clearly the mass function of $\mathbf{I}|\boldsymbol{\theta}$ satisfies

$$\begin{aligned} p(\mathbf{i}|\boldsymbol{\theta}) &= p(i_1|\boldsymbol{\theta})p(i_2, \ldots, i_n|\boldsymbol{\theta}, i_1) \\ &= \theta_{i_1} p(i_2, \ldots, i_n|\boldsymbol{\theta}, i_1) \end{aligned}$$

where $\mathbf{i} = i_1, \ldots, i_n$. And if $\mathbf{r}$ is sufficent for $\boldsymbol{\theta}$, as in the Dirichlet model,

$$p(\mathbf{i}|\boldsymbol{\theta}) = \theta_{i_1} p(i_2, \ldots, i_n|i_1) \tag{6.1.1}$$

If we now choose a distribution on the order statistic $\mathbf{Y}$ such that

$$\mathbf{Y} \perp\!\!\!\perp (\boldsymbol{\theta}, \mathbf{I}) \tag{6.1.2}$$

this will give us an induced distribution on $\mathbf{X}$ from which we can learn about $\boldsymbol{\theta}$ only through $\mathbf{I}$. On the other hand if, conditional on $\mathbf{I}$, $\mathbf{Y}$ is dependent on $\boldsymbol{\theta}$ then more might be learned about $\boldsymbol{\theta}$ by learning about some component of $\mathbf{X}$.

So it is clear that if we always learn about the total ordering $\mathbf{I}$ of the competing company's bids a model satisfying (6.1.2) will allow us to legitimately discard all other information about the actual value of bids as irrelevant to our future forecasts. And if in addition (6.1.1) holds and $\boldsymbol{\theta} \perp\!\!\!\perp (I_2, \ldots, I_n) \mid I_1$ the identity of the winner of the contract is all that is needed to fully update our distribution.

Is it possible to go further than this ? In practice it is rare for any company to be given the value of $\mathbf{I}$. For example, he might only learn that he did not win, or the identity of the winner, or the identity of the winner and his own position. Clearly given only a non-monotone function of $\mathbf{I}$, the actual value of bids may need to be used to update the distribution on $\boldsymbol{\theta}$. Usually each company F_i will at least know the value of its own bid X_i. So a minimal requirement that would allow us only to use functions of order statistics to update our beliefs is that $\mathbf{X}$ be ***order independent*** (o.i.) – that is, for any interval A the event $X_i \in A$ is independent of $\mathbf{I}$, $1 \leq i \leq n$.

If $\mathbf{X}$ is o.i. with the additional information about the value of their own bid, each firm using an o.i. distribution should update the the vector $\boldsymbol{\theta}$ only through the information they learn about $\mathbf{I}$. We now show that o.i. is not a property of of any non-degenerate countably additive distribution on $\mathbf{X}$. Define $I(i,j)$ as

$$I(i,j) = \begin{cases} 1 & X_i > X_j \\ 0 & \text{otherwise} \end{cases} \tag{6.1.3}$$

and assume that the joint distribution across X_i, X_j satisfies $P(X_i < X_j) < 1$ and $P(X_i > X_j) < 1$. This simply precludes the case where the supports of X_i and X_j do not overlap, since then the distribution across X_i, X_j is trivialy order independent. We now have;

THEOREM 6.1.4.

If, for some pair i, j $(1 \leq i \neq j \leq n)$, all the events of the form $A_x = \{X_i > x\}$ and $A'_x = \{X_j > x\}$ are independent of $I(i,j)$, and $P(X_i = X_j) = 0$, then the distribution across $X_1, \ldots, X_n$ cannot be countably additive.

PROOF:

$P(X_i > x \mid I(i,j) = 1) = P(X_i > x \mid X_i > X_j)$

$$= P(X_i > x \mid X_i > X_j > x)P(X_j > x) + P(X_i > x \mid X_i > X_j = x)P(X_j = x)$$
$$+ P(X_i > x \mid X_i > x > X_j)P(X_i > x > X_j) + P(X_i > x \mid x \geq X_i > X_j)P(x \geq X_i)$$

$$= P(X_j > x) + P(X_j = x) + P(X_i > x > X_j)$$

But since $A_x \perp\!\!\!\perp I(i,j)$, $P(X_i > x \mid I(i,j) = 1) = P(X_i > x)$ thus,

$$P(X_i > x) = P(X_j > x) + P(X_j = x) + P(X_i > x > X_j) \tag{6.1.5}$$

Permuting i and j in the above, conditioning on $I(i,j) = 0$ and using that $A'_x \perp\!\!\!\perp I(i,j)$ gives

$$P(X_j > x) = P(X_i > x) + P(X_i = x) + P(X_j > x > X_i) \tag{6.1.6}$$

Adding (6.1.5) and (6.1.6) and using the positivity of probability we find

$$P(X_i > x > X_j) = P(X_j > x > X_i) = 0 \qquad \text{for all } x \in \mathbb{R} \tag{6.1.7}$$

and

$$P(X_i = x) = P(X_j = x) = 0 \qquad \text{for all } x \in \mathbb{R}$$

Now since $P(X_i = X_j) = 0$ we can, without loss of generality, assume that

$$P(X_i < X_j) > 0 \tag{6.1.8}$$

If the (X_i, X_j) are countably additive then

$$P(X_i < X_j) \leq \sum_{B_x \in \mathcal{B}} P((X_i, X_j) \in B_x) \tag{6.1.9}$$

where $\mathcal{B} = \{B_x\}$ is any countable open covering of $\{(x_i, x_j) | x_i < x_j\}$. Well if

$$B_x = \{(x_i, x_j) \,|\, x_i < x < x_j\}$$

then $\mathcal{B} = \{B_x \,|\, x \text{ rational }\}$ forms such an open covering, since the rationals are dense in $\mathbb{R}$. But from (6.1.7)

$$P((X_i, X_j) \in B_x) = P(X_i < x < X_j) = 0$$

thus from (6.1.9)

$$P(X_i < X_j) = 0$$

this contradicts our assumption (6.1.8). It thus follows that X_i, X_j cannot be jointly countably additive, hence $X_1, \ldots, X_n$ cannot be jointly countably additive. □

So order independence is not a property of countably additive distributions. However De Finetti (1974), Heath & Sudderth (1978), Sudderth (1981) and Fishburn (1986) have all argued strongly that there is no convincing logical argument that Bayesian models should exhibit countably additive distributions on observables. Finitely additive priors are often used in an analysis and can be justified both theoretically, as models exhibiting certain forms of invariance, see Eaton (1982), and pragmatically, as close approximations to countably additive distributions which are computationally messy.

6.2 Characterising Order Independent Distributions.

In this section we construct a finitely additive joint distribution across the random variables $X_1, \ldots, X_n$, which is order independent, and hence establish that such a distribution can exist. We then go on to partially characterise the class of o.i. distributions.

Let $\mathbf{Y} = (Y_1, \ldots, Y_n)$ and define Y_i by

$$Y_i = Y_1 + \sum_{j=2}^{i} \epsilon_j \qquad \text{for } i \geq 2$$

where Y_1 has an arbitary left continuous distribution, and $\epsilon_2, \ldots, \epsilon_n$ are identically distributed positive random variables each with a distribution satisfying

$$P(\epsilon_i \in A) = \begin{cases} 1 & \text{if } A \subseteq (0, \eta) \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } all\ \eta > 0 \qquad (6.2.1)$$

Note that the distribution function of ϵ_i is not right continuous at zero. It thus follows that ϵ_i is not countably additive, $1 \leq i \leq n$.

Consider now the random variable $\mathbf{X} = (X_1, \ldots, X_n)$ defined by demanding that its order statistic will be $\mathbf{Y}$, i.e. $Y_i = X_{(i)}$, together with some distribution on its ordering $\mathbf{I}$.

We now prove a series of results that allow us to deduce that the distribution implied on $\mathbf{X}$ has the property of order independence.

LEMMA 6.2.2.

If $\epsilon^t = \sum_{i=2}^{t} \epsilon_i$, then ϵ^t has a distribution satisfying (6.2.1) for all t.

PROOF:

From (6.2.1) we know that for all i

$$P(\epsilon_i \in (0, \frac{\eta}{t-1})) = 1 \qquad \text{for all } \eta > 0$$

Thus

$$P(\sum_{i=2}^{t} \epsilon_i \in (0, \sum_{i=2}^{t} \frac{\eta}{t-1})) = P(\epsilon^t \in (0, \eta)) = 1 \qquad \text{for all } \eta > 0$$

so ϵ^t also has the property (6.2.1). $\square$

LEMMA 6.2.3.

If Z is a random variable with left continuous distribution function and ϵ has the property (6.2.1), then for any interval A,

$$P(Z + \epsilon \in A) = P(Z \in A)$$

PROOF:

Since the distribution of Z is left continuous it is sufficent to prove that

$$P(Z \in [a, b)) = P(Z + \epsilon \in [a, b))$$

for any finite interval $[a, b)$. Clearly, since $\epsilon > 0$,

$$P(Z < b) \geq P(Z + \epsilon < b) \tag{6.2.4}$$

and from (6.2.1)

$$P(Z + \epsilon < b) \geq P(Z < b - \eta) \qquad \text{for all } \eta > 0 \tag{6.2.5}$$

that is for all $\eta > 0$

$$P(Z < b - \eta) \leq P(Z + \epsilon < b) \leq P(Z < b)$$

which implies $P(Z + \epsilon < b) = P(z < b)$

Similarly, since $\epsilon > 0$

$$P(Z > a) \leq P(Z + \epsilon > a)$$

and from (6.2.1)

$$P(Z + \epsilon > a) \leq P(Z > a - \eta) \qquad \text{for all } \eta > 0$$

that is, for all $\eta > 0$

$$P(Z > a) \leq P(Z + \epsilon > a) \leq P(Z > a - \eta)$$

which implies $P(Z + \epsilon > a) = P(z > a)$.

Also, since the distribution of Z is left continuous, it follows that $P(Z = a) = P(Z + \epsilon = a) = 0$.

Hence the result follows. $\square$

Note that if Z is countably additive the left continuity condition means Z must be continuous. If Z is discrete, the distribution of $Z + \epsilon$ is different from Z in that its distribution function fails to be right continuous at the points with probability mass on them. This also means that the distribution of $Z + \epsilon$ is not countably additive.

Lemma 6.2.3 shows us that, provided Y_1 is left continuous, $Y_1, Y_2, \ldots, Y_n$ all have the same probability of lying in any interval A. This in turn means that $X_1, \ldots, X_n$ each assign the same probability to lying in A. Thus we can say

$$\begin{aligned} P(X_i \in A) = P(X_i \in A \mid X_i = Y_j) = P(X_i \in A \mid I) & \qquad \text{for all } i, j \\ = P(Y_j \in A \mid X_i = Y_j) = P(Y_j \in A) & \qquad (6.2.6) \end{aligned}$$

since by the construction (6.1.2) $\mathbf{Y} \perp\!\!\!\perp \mathbf{I}$. Since (6.2.6) holds for *all* intervals A, we have constructed an o.i. distribution on $\mathbf{X}$.

Before continuing we note the following definitions concerning types of finitely additive distributions.

Let $\pi(.)$ be an unconditional probability defined on a sample space Ω, and let $H = \{H_1, H_2, \dots\}$ be a countable partition of Ω. $\pi(.)$ is said to be ***purely finitely additive*** if, for all $\epsilon > 0$, there exists a partition, H, of Ω such that

$$\sum_{i=1}^{\infty} \pi(H_i) < \epsilon$$

Furthermore $\pi(.)$ is said to be ***strongly finitely additive*** if there exists a partition, H, such that

$$\pi(H_i) = 0 \qquad \text{for all } i$$

If we assume $P(X_i = X_j) = 0$ for all i, j, $i \neq j$, then we have the following

THEOREM 6.2.7.

If $\mathbf{X}$ satisfies the conditions of Theorem 6.1.4 then the distribution of $\mathbf{X}$ is strongly finitely additive

PROOF:

Define $B_x(i,j)$ as follows

$$B_x(i,j) = \{x_1, \dots, x_n | x_i < x < x_j\}$$

and B_x as

$$B_x = \bigcup_{1 \leq i \neq j \leq n} B_x(i,j)$$

then clearly $C = \{B_x | x \text{ rational }\}$ forms an open covering of $\mathbb{R}/\{x_1, \dots, x_n | x_1 = \dots = x_n\}$. Also

$$P(B_x) \leq \sum_{1 \leq i \neq j \leq n} P(B_x(i,j)) = 0$$

since $X_i, X_j \perp\!\!\!\perp I(i,j)$ implies $P(B_x(i,j)) = 0$ as in Theorem 6.1.4.

Thus the covering C has zero probability, it therefore follows that there exists a partition of $\mathbb{R}/\{x_1, \dots, x_n | x_1 = \dots = x_n\}$ with zero probability. Hence, since $P(X_i = X_j) = 0$ for all i, j, the distribution on $X_1, \dots, X_n$ is strongly finitely additive. $\square$

Clearly this result implies the strong finite additivity of o.i. distributions.

We now show that the property (6.2.1) of finitely additive distributions is *neccessary* for o.i. if all random variables are bounded above and below.

LEMMA 6.2.8.

Let $\epsilon_{ij} = X_i - X_j$, where X_i, X_j satisfy the conditions of theorem 6.1.4, then, if there exists α, β such that $P(\alpha \le X_k \le \beta) = 1$ for all k, $|\epsilon_{ij}|$ satisfies (6.2.1).

PROOF:

As in the proof of Theorem 6.1.4 define the region B_x as

$$B_x = \{ (x_i, x_j) \mid x_i < x < x_j \}$$

and define the set $B(\delta)$ as follows

$$B(\delta) = \bigcup_{k=0}^{m} B_{\alpha + k\delta} \qquad \alpha + m\delta \ge \beta$$

Then $B(\delta)$ covers the region $|\epsilon_{ij}| > \delta$, thus

$$P(|\epsilon_{ij}| > \delta) \le P(B(\delta)) \le \sum_{k=0}^{m} P(B_{\alpha+k\delta}) = 0$$

since from Theorem 6.1.4 $P(B_x) = 0$ for all $x \in \mathbb{R}$.

Since this is true for any $\delta > 0$ we have

$$P(|\epsilon_{ij}| > \delta) = 0 \qquad \text{for all } \delta > 0$$

which tells us that $|\epsilon_{ij}|$ has the property (6.2.1) □

The first consequence of this result is that it characterises o.i. distributions on finite ranges. That is (6.2.1) is a neccessary and sufficient condition for a sequence of random variables, defined on a finite range, to be order independent. Another consequence arises from a result of Lane & Sudderth (1979), as we now discuss. A sequence of random variables $X_1, X_2, \ldots$ is said to satisfy A_n if given $X_1, \ldots, X_n$, a further observation X_{n+1} is equally likely to lie in any of the $n+1$ intervals $J_i = (X_{(i)}, X_{(i+1)})$ $i = 0, \ldots, n$, where $X_{(0)} = -\infty$ and $X_{(n+1)}|X_1, \ldots, X_n = +\infty$, that is

$$P(X_{n+1} \in J_i) = \frac{1}{n+1} \qquad i = 0, \ldots, n$$

Hill (1988) provides a good review of the A_n property.

Let H_n be a class of measures β, defined on $X_1, \ldots, X_n$, which satisfy the following properties

(1) $X_1, \ldots, X_n$ are exchangeable under β

(2) $\beta(X_i = X_j) = 0$ for all i, j, $i \neq j$

(3) $X_1, \ldots, X_n$ have the A_n property under β

Hill (1968) has proved that H_n contains no countably additive measures. Lane & Sudderth (1979) have firstly proved that H_n is non–empty, and secondly that if $X_1, \ldots, X_n$ are bounded as in lemma 6.2.8

$$\beta(|X_{(n)} - X_{(1)}| > \epsilon) = 0 \qquad \text{for all } \epsilon > 0$$

It immediately follows from lemma 6.2.8 that H_n is a subset of the class of o.i. measures defined on bounded random variables. Or, loosely speaking, o.i. distributions on bounded random variables are a generalisation of exchangeable A_n distributions on bounded random variables.

Although lemma 6.2.8 concerned only sets of bounded variables its implications extend to results on all o.i. distributions by the next result.

LEMMA 6.2.9.

Let $\mathbf{Z} = (Z_1, \ldots, Z_n)$ where

$$Z_i = f(X_i) \qquad 1 \leq i \leq n$$

and f is an arbitrary strictly increasing function. Then if $\mathbf{X}$ is o.i. so is $\mathbf{Z}$.

PROOF:

Let $\mathbf{I}(\mathbf{X})$ and $\mathbf{I}(\mathbf{Z})$ indicate the order of $\mathbf{X}$ and $\mathbf{Z}$ respectively. Then since f is strictly increasing

$$\mathbf{I}(\mathbf{X}) = \mathbf{I}(\mathbf{Z})$$

Furthermore, noting that if A is an interval then so is $f(A)$

$$\{X_i \in A\} \amalg \mathbf{I}(\mathbf{X}) \qquad \text{for all intervals } A$$

$$\Longrightarrow$$

$$\{Z_i \in f(A)\} \amalg \mathbf{I}(\mathbf{Z}) \qquad \text{for all intervals } f(A)$$

Since any interval, B, of Z_i can be written as $B = f(A)$ for some A, by our construction of $\mathbf{Z}$. It follows that $\mathbf{Z}$ is o.i. $\square$

An o.i. distribution with variables X_i defined on the whole real line can be constructed as the image of an o.i. distribution with variables Z_i lying on $(0,1)$. For example set

$$Z_i = (1 + e^{-X_i})^{-1}$$

In general provided there exists a strictly increasing bounded function f from X_i to Z_i, $\mathbf{X}$ will be o.i. if and only if $|\epsilon_{ij}| = |Z_i - Z_j|$ exhibit the property (6.2.1) for $1 \le i \ne j \le n$.

We shall now construct an o.i. distribution which is not a simple extension of the A_n family.

Lemma 6.2.10.

Let W have the finitely additive distribution for which

$$P(W < a) = p \qquad \text{for all } a \in \mathbb{R}$$

$$P(W > a) = 1 - p \qquad \text{for all } a \in \mathbb{R}$$

and Z have any countably additive distribution. If J is a finite union of intervals, I is a measurable set and $W \amalg Z$ then

(1) $P(W + Z \in J) = P(W \in J)$

(2) $P(W + Z \in J \cap Z \in I) = P(W + Z \in J)P(Z \in I)$

that is

$$\{W + Z \in J\} \amalg \{Z \in I\}$$

Proof:

Since W is finitely additive and J is a finite union of intervals it is sufficient to prove the result for the case when J is a single interval. So let

$$a = \inf J \quad \text{and} \quad b = \sup J$$

where it is possible that $a = -\infty$ or $b = +\infty$.

Result (1) follows immediately from a simple convolution, and we have

$$P(W + Z \in J) = P(W \in J) = \begin{cases} 0 & a, b \text{ finite} \\ p & a = -\infty,\ b \text{ finite} \\ 1 - p & a \text{ finite},\ b = +\infty \end{cases}$$

The cases when a or b are infinite follow from the definition of W. When a and b are finite it can be seen that $P(W \in J) = 0$ as follows. Clearly

$$1 = P(W \in \mathbb{R}) = P(W < a \cup W \in (a,b) \cup W > b)$$

thus from the finite additivity of W

$$1 = p + P(W \in (a, b)) + 1 - p = 1 + P(W \in (a, b))$$

so $P(W \in (a, b)) = 0$ for a, b finite.

For (2) assume first that I is a finite interval with

$$c = \inf I \quad \text{and} \quad d = \sup I$$

Case I: a, b both finite. Immediately we have

$$P(W + Z \in J \cap Z \in I) = P(W + Z \in J)P(Z \in I)$$

since both sides are equal to 0.

Case II: $a = -\infty$, b finite.

$$P(W + d \in J \cap Z \in I) \leq P(W + Z \in J \cap Z \in I) \leq P(W + c \in J \cap Z \in I)$$

but $P(W + d \in J \cap Z \in I) = P(W + c \in J \cap Z \in I) = pP(Z \in I)$ since $W \amalg Z$, thus

$$P(W + Z \in J \cap Z \in I) = pP(Z \in I) = P(W + Z \in J)P(Z \in I)$$

since $P(W + Z \in J) = p$

Case III: a finite, $b = +\infty$.

$$P(W + d \in J \cap Z \in I) \leq P(W + Z \in J \cap Z \in I) \leq P(W + c \in J \cap Z \in I)$$

but $P(W + d \in J \cap Z \in I) = P(W + c \in J \cap Z \in I) = (1 - p)P(Z \in I)$ since $W \amalg Z$, thus

$$P(W + Z \in J \cap Z \in I) = (1 - p)P(Z \in I) = P(W + Z \in J)P(Z \in I)$$

since $P(W + Z \in J) = 1 - p$

So for any finite interval I, and J as defined above

$$\{W + Z \in J\} \amalg \{Z \in I\}$$

The countable additivity of Z now guarantees result (2), since any measurable set I can be constructed as a countable union of finite intervals. $\square$

We now construct a class of *shift invariant* o.i. distributions. Define

$$\mathbf{X} = \mathbf{Z} + W(1,1,\dots,1)^{\mathrm{T}} \tag{6.2.11}$$

Where $\mathbf{X} = (X_1,\dots,X_n)$, $\mathbf{Z} = (Z_1,\dots,Z_n)$, W has a distribution satisfying the conditions of Lemma 6.2.10 and we assume $\mathbf{Z} \perp\!\!\!\perp W$. It is easily seen that $\mathbf{X}$ is o.i. whenever $\mathbf{Z}$ has a countably additive distribution. For clearly by (6.2.11)

$$\{X_i \in A\} \perp\!\!\!\perp \mathbf{Z}\backslash\{Z_i\} \mid W, Z_i$$

and by Lemma 6.2.10 and (6.2.11)

$$\{X_i \in A\} \perp\!\!\!\perp Z_i \tag{6.2.12}$$

Also, since $W \perp\!\!\!\perp \mathbf{Z}$

$$W \perp\!\!\!\perp \mathbf{Z}\backslash\{Z_i\} \mid Z_i$$

The above imply

$$(\{X_i \in A\}, W) \perp\!\!\!\perp \mathbf{Z}\backslash\{Z_i\} \mid Z_i$$

which in turn implies

$$\{X_i \in A\} \perp\!\!\!\perp \mathbf{Z}\backslash\{Z_i\} \mid Z_i$$

This and (6.2.12) now tell us that

$$\{X_i \in A\} \perp\!\!\!\perp \mathbf{Z}$$

Thus, since $\mathbf{I}$ is only a function of $\mathbf{X}$ through $\mathbf{Z}$ in this model, and the choice of i was arbitary

$$\{X_i \in A\} \perp\!\!\!\perp \mathbf{I} \qquad 1 \le i \le n$$

showing that $\mathbf{X}$ is o.i.

One consequence of this result is that it illustrates that exchangeable sets of o.i. variables need not have the A_n property. As noted earlier Lane & Sudderth (1979) prove that all A_n distributions have the property that the range statistic on the n variables does not have a countably additive distribution. The $\mathbf{X}$ defined in (6.2.11) has a range statistic which *is* countably additive, since $X_{(n)} - X_{(1)} = Z_{(n)} - Z_{(1)}$ and $\mathbf{Z}$ is countably additive

We might now ask whether models with the o.i. property are useful ? From a pragmatic point of view we already know that o.i. distributions are useful. The set of models outlined above contains models whose forecasts agree with those of the Gates model, which are known to compare favourably with their competitors. Furthermore it is common for Bayesian models to have vague priors on a location variable and we have shown that shift invariant models have the required o.i. property.

It can also be argued that, in many cases, models with the o.i. property provide at least a reasonable approximation to realistic dynamics in contract bidding. For example, a shift invariant distribution can be justified as a good model for an observer, and sometimes even a decision maker, if he gauges that the magnitude of the effect of the decision makers action, i.e. fixing an appropriate markup, is liable to be swamped by the inherent inaccuracy of his cost estimates. It is also possible to construct decision mechanisms which directly give rise to o.i. models, an example is as follows.

Example.

A firm issues a contract. One company, F_i, out of n is then invited to bid with probability ϕ_i, $\sum_{i=1}^{n} \phi_i = 1$, having been *told* the lowest bid to date. This is done repeatedly until the customer chooses to stop the process. The company F_i knows that if no more bids are invited he will win as long as his bid is the lowest. So provided he judges that the probability p_t, the probability that the customer stops inviting bids at this stage t, is independent of the bid he makes he should let his bid, b_t, be less than the lowest bid to date, b_{t-1}, by a very small amount Δ_t, so

$$b_t = b_{t-1} - \Delta_t \qquad \Delta_t > 0$$

Suppose $s_t = \sum_{i=1}^{n} \Delta_i$ is convergent. Then it is clear that the distribution of order statisitics $\mathbf{Y}$ is as given in this section with Y_1 having the distribution (possibly unknown to an external observer) of $\lim_{t\to\infty} b_t$.

The type of auction described above does not occur in practice. However when competitors know each other well they may perceive this process in their minds – with their final bid the result of an infinite regress paralleling the auction we have described above.

In such ways it is possible to justify the use of o.i. distributions by an observer of the results of a competitive bidding process, whether he is inside one of the competing firms and knows their bid or whether he knows no bids. It should be noted however that they are unsuitable models for all but the impotent decision maker. However the models in chapter 4 show that modification of o.i. models to ones which allow control are possible.

7. THE TENDERER'S DECISION PROBLEM

In this chapter we take the viewpoint of the tendering organisation. It is assumed throughout that the contract is awarded to the lowest bidder. The problem is formulated in the next section and related to the well known Multi Armed Bandit Problem (MABP). The MABP is then reviewed and an extension of the solution concepts, for the MABP, to the tenderers problem is given. The main conclusion is that the tenderers problem, as formulated in §7.1, will require simplification in order to be solved fully and used as a base for a practical model for the tenderer

7.1 Formulation of the problem.

As described in chapter 1 the tenderer's problem is, at each time period, to invite r companies from the n possible, to submit a bid on the current contract, whence he incurs a cost equal to the minimum of the bids made by the r companies. An important point to note is that the value of r is fixed, and that the tenderer has no choice as to its value. This is because, in practice, organisations have firm guidelines as to the number of companies who are to be invited to bid, of course without this constraint the tenderer would simply invite as many companies as possible to bid. The choice of the companies must be made to minimise the expected total cost, or discounted cost, up to some finite, or infinite, horizon. The dilemma facing the tenderer is to strike a balance between choosing, on the one hand, the r companies to minimise immediate expected cost, and, on the other, to include little known firms in order to learn about their potential for making low bids in the future. For the time being we shall assume that the sequence of contracts are identical, this condition can then be dropped later as in chapter 5. We now formulate this problem mathematically.

Let $\mathbf{X}(t) = X_1(t), \dots, X_n(t)$ be a random vector representing the bids that would be made on the contract at time t by companies $F_1, \dots, F_n$ respectively. Assume $X_i(t)$ has a distribution function $F_i(t)$ which is parameterised by the 'state' vector $s_i(t)$. At each time period the tenderer must select a subset of $\mathbf{X}(t)$, $\mathbf{X}_\mathbf{i}(t) = X_{i_1}(t), \dots, X_{i_r}(t)$ say, whence he incurs cost

$$R_\mathbf{i}(\mathbf{s}_\mathbf{i}(t)) = min\,[X_{i_1}(t), \ \dots, X_{i_r}(t)]$$

where $\mathbf{s}_\mathbf{i}(t) = s_{i_1}(t), \dots, s_{i_r}(t)$. After observing $\mathbf{X}_\mathbf{i}(t)$ the states $s_1(t), \dots, s_n(t)$ are updated to reflect what the tenderer has learned about $F_1, \dots, F_n$. Thus the tenderers decision space D is given by

$$D = \{\, \mathbf{i} \,|\, \mathbf{i} = i_1, \dots, i_r \text{ is a combination of } r \text{ integers out of } 1, \dots, n \,\}$$

Define $u_i(t)$ as the control function at time t, i.e.

$$u_i(t) = \begin{cases} 1 & \text{if decision is } i \text{ at time } t \\ 0 & \text{otherwise} \end{cases}$$

So, if we are using geometric discounting with an infinite horizon, the problem is to sequentially choose control functions $u_i(t)$, $t = 0, 1, \ldots$, to minimise

$$\mathbf{E}\left[\sum_{i \in \mathcal{D}} \sum_{t=0}^{\infty} \lambda^t R_i(\mathbf{s}_i(t))\, u_i(t)\right]$$

It can now be seen that this problem is a generalisation of the Multi Armed Bandit Problem (MABP). The MABP was essentially solved by Gittins & Jones (1974), see also Gittins (1979). Whittle (1980) provided a more structured and pleasing proof of Gittins' results. We now provide a brief review of the Multi Armed Bandit problem, Gittins' solution and Whittle's proof.

7.2 Review of the Multi Armed Bandit Problem and its solution.

Let $\mathbf{X}(t) = X_1(t), \ldots, X_n(t)$ be a random vector, where $X_i(t)$ represents the value of a random process i at time t, $t = 0, 1, \ldots$. Exactly as in our problem let $F_i(t)$ be the distribution function of $X_i(t)$ and let $s_i(t)$ parameterise $F_i(t)$. If we define the reward to be equal to minus the cost in our problem, and restate the problem as maximising the expected total discounted reward, then the MABP is equivalent to our problem with the following restrictions:

(1) $X_1(t), \ldots, X_n(t)$ are mutually independent for all t

(2) only one process, j say, is to be observed at each time period, i.e. $r = 1$ and $\mathcal{D} = \{j \mid j = 1, \ldots, n\}$

(3) the updating on states must satisfy $s_i(t+1) = s_i(t)$ for $i \neq j$, i.e. the states of unobserved processes must remain unchanged

Gittins & Jones (1974) proved that for each proccess, i, we can compute an index, $G_i(s_i(t))$, such that it is optimal to observe, or continue, the process having the largest index. $G_i(s_i(t))$, the 'Gittins index', can be defined by

$$G_i(s_i(t)) = \sup_{\tau > 0}\left[\sum_{l=0}^{\tau-1} \frac{\lambda^l R_i^*(s_i(t+l))}{1 - \lambda^\tau}\right]$$

where

$$R_i^*(s_i(t+l)) = \mathbf{E}[R_i(s_i(t+l)) \mid X_i \text{ observed at times } t+1, t+2, \ldots, t+l-1]$$

The crucial points about this solution are, firstly, that the index for process i, G_i, is only a function of the state of process i, s_i. And, secondly, the optimal decision can be ascertained simply by forecasting ahead each individual process.

A pleasing interpretation of the Gittins index is as follows.

Consider an individual process, X_i say, and imagine a simplified bandit problem involving processes X_i and X^M, where X^M offers a constant known reward M at every time period, i.e. at each time period the problem is to choose between observing the random process X_i or opting for a known reward M.

Once again assuming geometric discounting and an infinite horizon, the optimality equation for choosing between X_i and X^M at time t is

$$\phi_i(s_i(t), M) = \max\{\, M + \lambda\phi_i(s_i(t), M)\,,\; R_i(s_i(t)\,) + \lambda \mathrm{E}[\phi_i(s_i(t+1), M)\,]\,\} \tag{7.2.1}$$

Where $\phi_i(s_i(t), M)$ is the optimal total expected discounted reward, when $X_i(t)$ is in state $s_i(t)$, and the expectation is with respect to the state transition $s_i(t) \to s_i(t+1)$, which is in turn dependent on the observation $X_i(t)$.

Now clearly, if it is optimal to observe X^M for the first time at $t = \tau_i$, then, since the state of $X_i(t)$ will remain unchanged, it will be optimal to observe X^M at all following time periods $t = \tau_i + 1, \tau_i + 2, \ldots$. Thus we can rewrite (7.2.1) as

$$\phi_i(s_i(t), M) = \max\{\, M/(1-\lambda)\,,\; R_i(s_i(t)\,) + \lambda \mathrm{E}[\phi_i(s_i(t+1), M)\,]\,\} \tag{7.2.2}$$

which is equivalent to having the option to retire, from continuing X_i, at each time period and collect terminal reward $M/(1-\lambda)$.

Hence the optimal policy is of the form, 'continue X_i until some time period τ_i and then retire and collect $M/(1-\lambda)$'. Thus

$$\phi_i(s_i(t), M) = \sum_{l=0}^{\tau_i - 1} \lambda^l R_i(s_i(t+l)\,) + \lambda^{\tau_i}\frac{M}{1-\lambda} \tag{7.2.3}$$

Now define $M_i = M_i(s_i(t)\,)$ as the value of M for which the options of continuing X_i or X^M are indistinguishable. Thus from (7.2.2) we have

$$\phi_i(s_i(t), M_i) = \frac{M_i}{1-\lambda} \tag{7.2.4}$$

Substituting (7.2.4) in to (7.2.3) yields

$$\frac{M_i}{1-\lambda} = \sum_{l=0}^{\tau_i - 1} \lambda^l R_i'(s_i(t+l)) + \lambda^{\tau_i} \frac{M_i}{1-\lambda}$$

solving for M_i implies

$$\frac{M_i}{1-\lambda} = \frac{\sum_{l=0}^{\tau_i-1} \lambda^l R_i'(s_i(t+l))}{1-\lambda^{\tau_i}}$$

since τ_i is the optimal retirement time we have

$$\frac{M_i}{1-\lambda} = \sup_{\tau>0} \left[\frac{\sum_{l=0}^{\tau-1} \lambda^l R_i'(s_i(t+l))}{1-\lambda^{\tau}} \right]$$

i.e. $M_i/1-\lambda$ is the Gittins index for process i. So the Gittins result says that if we compute the constant process equivalent to each random process, then it is optimal to continue the random process with the largest constant equivalent. It is common to refer to $M_i = (1-\lambda)G_i$ as the 'indifference value' for process i.

We now discuss one of the most pleasing proofs of the optimality of the Gittins index policy, due to Whittle (1980).

Whittle's approach was to use what is almost a standard technique for solving this type of stochastic control problem, namely

(1) restate the problem as an optimal stopping problem.

(2) 'guess' the optimal stopping rule, and compute the payoff under this rule.

(3) check whether this payoff satisfies the optimality equation.

First some notation and preliminaries. Assume

$$k \leq \frac{R_i(s_i)}{1-\lambda} \leq K \qquad \text{for all } i \text{ and } s_i \qquad (7.2.5)$$

Let $s(t) = (s_1(t), \ldots, s_n(t))$ and let $\Phi(s)$ be the optimal expected total discounted payoff for the MABP. So Φ satisfies the optimality equation

$$\Phi = \max_i L_i \Phi \qquad (7.2.6)$$

where the operator L_i is defined by

$$L_i \Phi(s(t)) = R_i(s_i(t)) + \lambda \mathrm{E}[\Phi(s(t+1)) \mid X_i \text{ observed at time } t\ ]$$

The restatement of the problem as an optimal stopping problem proceeds as follows. Consider the MABP as stated, with the added possibility of retiring at some stage to receive a terminal reward M. Let $F(\mathbf{s}, M)$ be the analogue of $\Phi(\mathbf{s})$ for this extended problem, so F satisfies the optimality equation

$$F = \max\{M\,,\, \max_i L_i F\} \tag{7.2.7}$$

The following two lemmas tell us about the form of F. Detailed proofs are omitted in this review but can be found in Whittle (1980).

LEMMA 1.

$F(\mathbf{s}, M)$ is a non–decreasing convex function of M, and

$$F(\mathbf{s}, M) = \begin{cases} \Phi(\mathbf{s}) & M \leq k \\ M & M \geq K \end{cases}$$

LEMMA 2.

For almost all M

$$\frac{\partial F(\mathbf{s}, M)}{\partial M} = E[\lambda^{T_M} | \mathbf{s}(0) = \mathbf{s}]$$

Where T_M is the optimal retirement time, and the expectation is with respect to T_M under the optimal policy.

Having reformulated the MABP as an optimal stopping problem Whittle's 'guessed' stopping rule is summarised in the conjecture below.

CONJECTURE.

The optimal policy is of the following form. When the state of an individual process enters a given set S, abandon the process. When all processes have been abandoned, retire and collect the terminal reward.

Conditional on this conjecture Whittle proved the following three lemmas. Note that $\phi_i(.,.)$, r_i and M_i are as defined earlier.

LEMMA 3.

$$E[\lambda^{T_M}] = \prod_i E[\lambda^{r_i}]$$

This lemma clearly relies on the independence of the individual processes.

LEMMA 4.

$$\frac{\partial F(\mathbf{s},M)}{\partial M} = \prod_i \frac{\partial \phi_i(s_i,M)}{\partial M}$$

Integrating this yields

LEMMA 5.

$$F(\mathbf{s},M) = K - \int_M^K \prod_i \frac{\partial \phi_i(s_i,m)}{\partial m}\,dm$$

If $F(\mathbf{s},M)$ is of the form in Lemma 5 we shall say it is of the 'Whittle form'.

Whittle now showed that this payoff is optimal, and hence that the conjectured policy is optimal, by verifying that under certain conditions $F(\mathbf{s},M)$, as in Lemma 5, satisfies the optimality equation (7.2.7). These conditions in turn establish the optimality of the Gittins index policy.

Let $M_{(i)} = \max_{j\neq i} M_j$ and define

$$p_i(\mathbf{s},M) = \prod_{j\neq i} \frac{\partial \phi_j(s_j,M)}{\partial M}$$

It can immediately be seen that $p_i(\mathbf{s},M) = 1$ if $M \geq M_{(i)}$. Since if $M \geq M_j$, $\phi_j(s_j,M) = M$ as the terminal reward available is greater than the indifference value. So if $M \geq M_j$ for all $j \neq i$ i.e. $M \geq M_{(i)}$ then $\frac{\partial \phi_j}{\partial M} = 1$ for all $j \neq i$, hence $p_i = 1$.

So if $F(\mathbf{s},M)$ is of the Whittle form integration by parts yields

$$\begin{aligned}
F(\mathbf{s},M) &= K - \int_M^K \prod_i \frac{\partial \phi_i(s_i,m)}{\partial m}\,dm \equiv K - \int_M^K p_i(\mathbf{s},M)\frac{\partial \phi_i(s_i,m)}{\partial m}\,dm \\
&= K - [p_i(\mathbf{s},m)\phi_i(\mathbf{s},m)]_{m=M}^K + \int_M^K \phi_i(\mathbf{s},M)\,dp_i(\mathbf{s},m) \\
&= K - (1.K - p_i(\mathbf{s},M)\phi_i(\mathbf{s},M)) + \int_M^K \phi_i(\mathbf{s},M)\,dp_i(\mathbf{s},m) \\
&= p_i(\mathbf{s},M)\phi_i(\mathbf{s},M) + \int_M^K \phi_i(\mathbf{s},M)\,dp_i(\mathbf{s},m) \qquad (7.2.8)
\end{aligned}$$

since $K \geq M_{(i)}$ implies $P_i(\mathbf{s},K) = 1$ and $\phi_i(s_i,K) = K$.

Suppressing the dependency on $\mathbf{s}$ and s_i, the term $F(M) - L_i F(M)$ is now formed as,

$$F(M) - L_i F(M) = p_i(M)\delta_i(M) + \int_M^K \delta_i(m)\,dp_i(m) \qquad (7.2.9)$$

where $\delta_i(m) = \phi_i(m) - L_i\phi_i(m)$. Clearly, since $\phi_i(m)$ satisfies $\phi_i(m) = \max(m, L_i\phi_i(m))$, it must be that $\delta_i(m) \geq 0$ for all m, which implies $F(m) - L_i F(m) \geq 0$ for all m.

Assume now that the following two conditions hold

(i) $M \leq M_i$

(ii) $M_i \geq M_{(i)}$

By (i) we must have $\phi_i(M) = L_i\phi_i(M)$ i.e. $\delta_i(M) = 0$, since the terminal reward M is less than the indifference value M_i, so immediate retirement cannot be optimal. Thus by (i) equation (2.2.5) becomes

$$F(M) - L_i F(M) = \int_{M_i}^{K} \delta_i(m)\, dp_i(m)$$

By (ii) we have that for all $m \in [M_i, K]$, $p_i(m) = 1$ and $\frac{\partial p_i(m)}{\partial m} = 0$, thus the above integral is equal to 0, and $F(M) - L_i F(M) = 0$.

We thus have that the optimal payoff is of the Whittle form if (i) and (ii) hold. But (i) just tells us that retirement is not optimal, and (ii) is equivalent to having $M_i = \max_j M_j$, or equivalently $G_i = \max_j G_j$. So the optimality of the Gittins index policy for the MABP is established.

We conclude this review section by mentioning some of the existing extensions to the MABP.

An important limitation on the optimality of a Gittins index policy was proved by Berry & Fristedt (1985). They proved that if, in the infinite horizon MABP, we do not have geometric discounting, then a Gittins index policy is not neccesarily optimal.

Nash (1980) considered a MABP where the reward was not just a function of the state of the observed process, but was a function of all the states, in a multiplicatively separable way i.e. the reward when observing process i was

$$R_i(s_i)\prod_{j\neq i} Q_j(s_j)$$

He then proved that an index policy is optimal, where the optimal index is given by

$$N_i(\mathbf{s}(t)) = \sup_{\tau\in T}\left[\frac{\sum_{l=0}^{\tau-1}\lambda^l R_i^l(s_i(t+l))}{Q_i(s_i(t)) - \lambda^\tau Q_i^\tau(s_i(t+\tau))}\right]$$

where $Q_i^\tau(s_i(t+\tau)) = \mathrm{E}[Q_i(s_i(t+\tau)) \mid$ process i observed at times $t, \ldots, t+\tau-1$]. And T is a restricted set of stopping times defined by

$$T = \begin{cases} T_i'(\mathbf{s}) & \text{if } T_i'(\mathbf{s}) \text{ is non empty} \\ T_i & \text{otherwise} \end{cases}$$

where T_i is the set of all strictly positive stopping times, and

$$T_i' = \{\tau \mid \tau \in Z \text{ and } Q_i(s_i) - \lambda^\tau Q_i'(s_i(t+\tau)) < 0\}$$

and is of course a subset of T_i.

Clearly $N_i(s)$ can take negative values, and a process with a negative index is always preferred to a process with a positive index. Thus the optimal process to continue is the process with the negative index that is largest in modulus, or, if no process has negative index, then simply the one with the largest index. Intuitively this is acceptable, since $N_i(s) < 0$ is equivalent to $Q_i(s_i) - \lambda^\tau Q_i'(s_i(t+\tau)) < 0$, at the value of τ achieving the supremum. That is, continuing process i produces an immediate positive reward, and actually improves the expected discounted contribution that process i will make to the total reward in future i.e. $\lambda^\tau Q_i'(s_i(t+\tau)) > Q_i(s_i)$.

Eick (1988) considers the 'delayed response bandit'. In this problem the reward for continuing a process, j, is not immediately observable, but will be observed at the t^{th} time period later with probability $\theta_j(1-\theta_j)^t$ for some parameter θ_j

Eick then proves that, at a particular time period, there exists an optimal index, provided either $\lambda < \frac{1}{2}$ or there are no outstanding observations at that time period.

7.3 Generalising The MABP.

Returning to the tenderer's problem as formulated in §7.1, we shall now view it as a MABP with the following generalisations

G1: Allow the states of unobserved processes to change in a deterministic way.

G2: Allow more than one process to be observed at each time period. The reward being a function of the states of all the observed proccesses.

G3: Allow processes to be dependent

Note that generalisation G1 will permit us to let the individual processes develop with the Simple Power Steady model discussed and used earlier.

In one sense these three generalisations can be viewed as a single generalisation that permits the states of unobserved processes to change in some random manner. Clearly G1 and G3 can be viewed as special cases of this generalisation, and G2 can be modelled as a special case of G3 by saying that we must observe one of the $\binom{n}{r}$ processes obtained by grouping all the possible combinations of r processes out of n.

We shall thus define the Generalised Multi Armed problem (GMABP), as a MABP which permits random changes to the states of unobserved processes.

As a first attempt to tackle the GMABP we shall investigate how far the the ideas underlying the Whittle proof can be extended. First, using the same notation as the previous section, we have the following

LEMMA 7.3.1.

If $R'_j(s_j(t+l))$ is a non decreasing function of l, then

$$\phi_j(s_j(t), M) = \max\{M, M_j(s_j(t))\}$$

furthermore

$$M_j(s_j(t)) = \sum_{l=0}^{\infty} \lambda^l R'_j(s_j(t+l))$$

PROOF:

First note that an equivalent statement of the form of ϕ_j is

$$\phi_j(s_j(t), M) = \begin{cases} M & M > M_j \\ M_j & M \leq M_j \end{cases}$$

$M_j = M_j(s_j(t))$

The crux of the lemma is that ϕ_j depends on M only through the immediate retirement option. Clearly this is only so when the optimal stopping time, τ, is equal to 0 or ∞, depending on whether $M > M_j$ or $M \leq M_j$ respectively.

Let $U_r = \sum_{l=0}^{r-1} \lambda^l R'_j(s_j(t+l)) + \lambda^r M$, thus $\phi_j(s_j(t), M) = \max_r U_r$.

So the optimal stopping time is at $r = 0$ iff $M \geq U_r$ for all r.

$$\Leftrightarrow \qquad M(1-\lambda^r) \geq \sum_{l=0}^{r-1} \lambda^l R'_j(s_j(t+l)) \qquad \text{for all } r$$

$$\Leftrightarrow \qquad M \geq \sup_r \left[\frac{\sum_{l=0}^{r-1} \lambda^l R'_j(s_j(t+l))}{1-\lambda^r} \right] \equiv M_j^{\text{upper}}$$

And the optimal stopping time is at $\tau = \infty$ iff $U_\infty \geq U_\tau$ for all τ.

$$\Leftrightarrow \qquad \sum_{l=0}^{\infty} \lambda^l R'_j(s_j(t+l)) \geq \sum_{l=0}^{\tau-1} \lambda^l R'_j(s_j(t+l)) + \lambda^\tau M \qquad \text{for all } \tau$$

$$\Leftrightarrow \qquad \lambda^\tau M \leq \sum_{l=\tau}^{\infty} \lambda^l R'_j(s_j(t+l)) \qquad \text{for all } \tau$$

$$\Leftrightarrow \qquad M \leq \inf_\tau \left[\sum_{l=0}^{\infty} \lambda^l R'_j(s_j(t+\tau+l)) \right] \equiv M_j^{\text{lower}}$$

So ϕ_j is of the specified form iff $M_j^{\text{lower}} = M_j^{\text{upper}} \equiv M_j$.

Let

$$W_\tau = \frac{\sum_{l=0}^{\tau-1} \lambda^l R'_j(s_j(t+l))}{1-\lambda^\tau} \quad \text{and} \quad W_\tau^* = \sum_{l=0}^{\infty} \lambda^l R'_j(s_j(t+\tau+l))$$

so

$$M_j^{\text{upper}} = \sup_\tau W_\tau \quad \text{and} \quad M_j^{\text{lower}} = \inf_\tau W_\tau^*$$

If we know that $R'_j(s_j(t+l))$ is a non–decreasing function of l, then a little algebra shows

$$W_{\tau+1} - W_\tau = \frac{(1-\lambda)\lambda^\tau}{(1-\lambda^{\tau+1})(1-\lambda^\tau)} \left[R'_j(s_j(t+\tau)) \sum_{l=0}^{\tau-1} \lambda^l - \sum_{l=0}^{\tau-1} \lambda^l R'_j(s_j(t+l)) \right] \geq 0$$

Thus $\sup_\tau W_\tau = W_\infty = \sum_{l=0}^{\infty} \lambda^l R'_j(s_j(t+l)) = M_j^{\text{upper}}$.

Similarly

$$W_{\tau+1}^* = \sum_{l=0}^{\infty} \lambda^l R'_j(s_j(t+\tau+l+1)) \geq \sum_{l=0}^{\infty} \lambda^l R'_j(s_j(t+\tau+l)) = W_\tau^*$$

Thus $\inf_\tau W_\tau^* = W_0^* = \sum_{l=0}^{\infty} \lambda^l R'_j(s_j(t+l)) = M_j^{\text{lower}}$

Therefore $M_j^{\text{lower}} = M_j^{\text{upper}} = \sum_{l=0}^{\infty} \lambda^l R'_j(s_j(t+l)) \equiv M_j$ □

And now;

THEOREM 7.3.2.

If $R'_j(s_j(t+l))$ is a non decreasing function of l, then the following are both true for the GMABP

(i) *The optimal payoff is of the Whittle form*

(ii) *The Gittins index policy is optimal* *if and only if*

$$E[\max_j M_j(s_j(t+1))] = E[\max\{M_{(i)}(\mathbf{s}(t));\ M_i(s_i(t+1))\}] \tag{$*$}$$

where process i is to be the observed process at time t, i.e. $M_i(s_i(t)) = \max_j M_j(s_j(t))$

Before the proof, we note that condition $(*)$ has the pleasing intuitive interpretation that it requires the expected value of the Gittins index at time $t+1$ to be the same, with the random updating of the states of the unobserved processes, as it would be if there was no updating of these states.

PROOF:

As in Whittles proof consider the GMABP reformulated as a stopping problem with the option to retire at any stage and receive terminal reward M.

For ease of notation we shall adopt the convention of only indicating dependency on time for time $t+1$ and not for time t. So, for example, we shall write $\phi_i(s_i(t), M)$ as $\phi_i(M)$ but $\phi_i(s_i(t+1), M)$ in full.

As in the Whittle proof define

$$p_i(M) = \prod_{j \neq i} \frac{\partial \phi_j(M)}{\partial M}$$

so from Lemma 7.3.1 we have

$$p_i(M) = \begin{cases} 1 & M > M_{(i)} \\ 0 & M \leq M_{(i)} \end{cases}$$

Where, as before, $M_{(i)} = \max_{(j \neq i)} M_j$. If we now take $F(M)$ to be the Whittle form of the payoff, then we want to find conditions to make $F - L_i F = 0$, when a Gittins index policy is being used. So we now form the term $F - L_i F$.

Recall Whittle shows that $F(M) = p_i(M)\phi_i(M) + \int_M^K \phi_i(m)\, dp_i(m)$ so

$$L_i F(M) = R_i + \lambda \mathrm{E}[p_i(M)\phi_i(M) + \int_M^K \phi_i(m)\, dp_i(m)] \tag{7.3.3}$$

Let $\Delta_i(\mathbf{s}(t+1), M) = p_i(\mathbf{s}(t+1), M) - p_i(M)$ and recall that $\lambda \mathrm{E}[\phi_i(s_i(t+1), M)] = L_i\phi_i(M) - R_i$ so (7.3.3) becomes

$$L_i F(M) = R_i + p_i(M)[L_i\phi_i(M) - R_i] + \int_M^K (L_i\phi_i(m) - R_i)\, dp_i(m) + \lambda \mathrm{E}[D_i(\mathbf{s}(t+1), M)]$$

where

$$D_i(\mathbf{s}(t+1), M) = \Delta_i(\mathbf{s}(t+1), M)\phi_i(s_i(t+1), M) + \int_M^K \phi_i(s_i(t+1), m)\, d\Delta_i(\mathbf{s}(t+1), m)$$

substituting $\delta(M) = \phi_i(M) - L_i\phi_i(M)$ gives

$$\begin{aligned}
&F(M) - L_i F(M) \\
&\quad = p_i(M)\delta(M) + R_i(p_i(M) - 1) + \int_M^K (\delta(m) + R_i)\, dp_i(m) - \lambda \mathrm{E}[D_i(\mathbf{s}(t+1), M)] \\
&\quad = p_i(M)\delta(M) + R_i(p_i(M) - 1) + \int_M^K \delta(m)\, dp_i(m) + R_i[p_i(m)]_M^K - \lambda \mathrm{E}[D_i(\mathbf{s}(t+1), M)] \\
&\quad = p_i(M)\delta(M) + \int_M^K \delta(m)\, dp_i(m) - \lambda \mathrm{E}[D_i(\mathbf{s}(t+1), M)]
\end{aligned}$$

This equation is completely analogous to (7.2.9) in Whittles proof. We know from Whittles proof that adoption of a Gittins index policy makes $p_i(M)\delta(M) + \int_M^K \delta(m)\, dp_i(m)$ equal to 0. Therefore, F is of the Whittle form and a Gittins index policy is optimal, if and only if $\mathrm{E}[D_i(\mathbf{s}(t+1), M)] = 0$.

Now $M_j = \sum_{l=0}^{\infty} \lambda^l R_j'(s_j(t+l)) \geq R_j'(s_j(t)) \sum_{l=0}^{\infty} \lambda^l = \frac{R_j'(s_j(t))}{1-\lambda} \geq k$,by initial assumption. If we now choose M such that $M \leq k$ we have the following:

(i) it will never be optimal to retire, since $M \leq k \leq M_j$.

(ii) $p_i(s, M) = 0$ for all states s.

Statement (i) is fine, as it simply means we are now just dealing with the GMABP, and (ii) means that the condition $\mathrm{E}[D_i(\mathbf{s}(t+1), M)] = 0$ can now be written as

$$\mathrm{E}[\int_k^K \phi_i(s_i(t+1), m)\, d\Delta_i(\mathbf{s}(t+1), m)] = 0$$

which, from the definition of $\Delta_i(\mathbf{s}(t+1), M)$, is equivalent to

$$\mathrm{E}[\int_k^K \phi_i(\mathbf{s}(t+1), m)\, dp_i(s_i(t+1), m)] = \mathrm{E}[\int_k^K \phi_i(s_i(t+1), m)\, dp_i(m)]$$

and from the form of $p_i(., M)$, is equivalent to

$$\mathrm{E}[\phi_i(s_i(t+1), M_{(i)}(\mathbf{s}(t+1)))] = \mathrm{E}[\phi_i(s_i(t+1), M(i))]$$

The result follows immediately from Lemma 7.3.1 . □

We might now ask how likely it is that this condition will be satisfied for MABP's with the individual generalisations G1, G2 and G3. Note that for a MABP with generalisation G1 only, condition (*) simplifies to either one of the following being true

(i) $M_j(s_j(t+1)) = M_j$ for all $j \neq i$

i.e. for the deterministic updating of unobserved states.

or

(ii) $M_i(s_i(t+1)) \geq M_{(i)}$ *and* $M_i(s_i(t+1)) \geq M_{[i]}(s(t+1))$

for all possible transitions of the state of the observed process.

An example is as follows.

Let the random variable $X_j(t)$ represent the value process j takes at time t, and assume $X_j(t)$ has p.d.f.

$$f_{X_j(t)}(x \mid p_t^j) = \begin{cases} p_t^j & X_j(t) = 0 \\ (1-p_t^j)g_j(x) & X_j(t) > 0 \end{cases} \tag{7.3.4}$$

where $g_j(x)$ is a known p.d.f. with bounded support. Now take the prior

$$p_t^j \sim Be(\alpha_t^j, \beta_t^j)$$

the Beta distribution with parameters α_t^j, β_t^j. Dropping the superscript j, this prior is equivalent to

$$\phi_t = \log\left(\frac{p_t}{1-p_t}\right) \sim LBe(\alpha_t, \beta_t)$$

the Logistic–Beta distribution with parameters α_t, β_t, which has p.d.f.

$$p(\phi_t \mid \alpha_t, \beta_t) = \frac{\Gamma(\alpha_t+\beta_t)}{\Gamma(\alpha_t)\Gamma(\beta_t)} \frac{e^{\alpha_t \phi_t}}{(1+e^{\phi_t})^{\alpha_t+\beta_t}}$$

Recall that the Logistic Beta and its multivariate analogue the Logistic Dirichlet were used extensively in chapter 4.

If we now let ϕ_t develop with the Simple Power Steady model with discount factor k, we have

$$\phi_{t+1} \mid \mathbf{x}^t \sim LBe(k\alpha_t, k\beta_t) \tag{7.3.5}$$

Where $\mathbf{x}^t$ are the observations from the particular process under consideration up to and including time t. An application of Bayes theorem yields

$$\phi_{t+1} \mid X(t+1), \mathbf{x}^t \sim \begin{cases} LBe(k\alpha_t + 1, k\beta_t) & X(t+1) = 0 \\ LBe(k\alpha_t, k\beta_t + 1) & X(t+1) > 0 \end{cases} \tag{7.3.6}$$

Providing the updating of the prior parameters α_t, β_t. The forecast distributions are given by;

LEMMA 7.3.7.

$$f_{X(t+s)|\mathbf{x}^t}(x\,|\,\mathbf{x}^t) = \begin{cases} p_{(t+s)} & X(t+s) = 0 \\ (1 - p_{(t+s)})g(x) & X(t+s) > 0 \end{cases}$$

where

$$p_{(t+s)} = E[p_{t+s}|\mathbf{x}^t] = \frac{\alpha_t}{\alpha_t + \beta_t} \qquad \text{for all } s$$

PROOF:

The result is immediate for $s = 1$, for $s > 1$ proceeding directly we have,

$$\begin{aligned} p_{(t+s)} &= P(X_{t+s} = 0\,|\,\mathbf{x}^t) \\ &= \int \cdots \int P(X_{t+s} = 0\,|\,\mathbf{x}^{t+s-1}) \prod_{i=1}^{s-1} f(X_{t+i}\,|\,\mathbf{x}^{t+i-1})\, dx_{t+1} \ldots dx_{t+s-1} \\ &= \sum_{r=0}^{s-2} \sum_{v \in C_r} q_t^r (1 - q_t)^{n-r-2} \pi(v, r) \end{aligned}$$

where $q_t = \alpha_t/\alpha_t + \beta_t$, C_r is the set of all combinations of r integers out of $1, \ldots, s-2$, and $\pi(v,r) = \alpha_{t+r}/\alpha_{t+r} + \beta_{t+r}$ given that a particular combination $v \in C_r$ of $x_{t+1}, \ldots, x_{t+s-1}$ are greater than 0.

Thus if $v = i_1, \ldots, i_r$ we have, from the updating (7.3.6),

$$\pi(v, r) = \frac{k^{s-2}\alpha_t + k^{s-i_1} + \cdots + k^{s-i_r}}{k^{s-2}(\alpha_t + \beta_t) + \sum_{l=0}^{s-3} k^l}$$

the above double sum can now be computed, yielding $p_{t+s} = \alpha_t/\alpha_t + \beta_t$.

An identical argument to the above yields

$$f_{X_{t+s}}(x\,|\,\mathbf{x}^t \cap X_{t+s} > 0) = (1 - \frac{\alpha_t}{\alpha_t + \beta_t})g(x)$$

and the result is proved. $\square$

This Lemma tells us that $R_j^t(s_j(t+l))$ is a non–decreasing function of l, (in fact it is a constant function of l), and also that

$$M_j(s_j(t)) = M_j(\alpha_t^j, \beta_t^j) = \frac{\beta_t^j m_j}{(\alpha_t^j + \beta_t^j)(1 - \lambda)} \qquad \text{for all } j$$

where $m_j = \int x g_j(x)\, dx$, is known.

The updating in (7.3.5) provides the deterministic updating of the states of the unobserved processes, i.e. $\alpha_{t+1} = k\alpha_t$, $\beta_{t+1} = k\beta_t$. Thus

$$M_j(\alpha^j_{t+1}, \beta^j_{t+1}) = \frac{k\beta^j_t m_j}{(k\alpha^j_t + k\beta^j_t)(1-\lambda)} = M_j(\alpha^j_t, \beta^j_t)$$

if process j is an unobserved process at time t. Hence, from the theorem, if we are observing one process, at each time period, from n processes developing independently with the above Simple Power Steady model, a Gittins index policy is optimal, and instructs us to observe the process with the largest value of

$$M_i(\alpha^i_t, \beta^i_t) = \frac{\beta^i_t m_i}{(\alpha^i_t + \beta^i_t)} \qquad i = 1, \ldots, n$$

Note that this is simply a one step look ahead policy.

It is worth mentioning that this example stemmed from an attempt to build a model for the tenderer based on 'spiked' distributions of the form (7.3.4). The logic behind this is that the tenderer will decide in his own mind a bid below which he will not award the contract, presumably because he would feel that quality would have to be compromised to produce such a low cost. On the assumption that the bidders are thinking along the same lines, we might model the bid distribution to be of the form (7.3.4) with the spike of probability p at this minimum bid and $g(x)$ decreasing in x, for example an exponential distribution. However the example above illustrates that even with specific distributional assumptions, such as these, solution to the GMABP is very difficult.

Although, as illustrated above, there are models for which theorem 7.3.2 is useful in determining a solution, the results in this section primarily serve to illustrate the difficulties faced in tackling complicated stochastic control problems such as the GMABP.

Without wishing to sound pessimistic, it is clear that if progress is to be made towards a practical model for the tenderer greater structure has to be incorporated in the formulation of the problem, leading to simpler stochastic control problems. This is one of the topics discussed in the next chapter. It should be stressed however that even simple well specified problems with finite time horizons can be computationally difficult and give results which are at first sight suprising, see for example Smith (1989 b).

8. FURTHER RESEARCH

As just mentioned at the end of chapter 7 one area very much in need of attention is the development of a practical model for the tenderer. An obvious way to approach this is to introduce greater structure into the problem. A simple model is as follows. Assume that bidders come from one of three classes C_1, C_2 and C_3, where, in terms of a random variable X representing the value of a bid made by a company in a given class, the classes are defined by

$$\begin{array}{lll} C_1 & : & X = 0 \text{ with probability } 1 \\ C_2 & : & X \text{ has known distribution function } F \\ C_3 & : & \begin{cases} X \in C_1 & \text{with probability } q \\ X \in C_2 & \text{with probability } 1-q \end{cases}, \; q \text{ known} \end{array}$$

and that within C_2 the bidders are exchangeable, say. These classes can be interpreted as follows. Class C_1 contains known reliable bidders. In the above definition we are rescaling this reliable bid to be 0. Class C_2 contains known but more erratic bidders, while C_3 contains companies about which we know very little. We also assume that when a company from C_3 is invited to bid it can then be classified as being in C_1 or C_2.

With this problem the tenderers decision space is given by

$$\{ (r_1, r_2, r_3) \mid r_i \in \mathbb{N} \text{ and } r_1 + r_2 + r_3 = r \}$$

where decision (r_1, r_2, r_3) corresponds to inviting r_i companies fron C_i to bid. The state at time t is $(s_1(t), s_2(t), s_3(t))$ where $s_i(t)$ is the number of companies in C_i at time t.

The advantage of this simple model is that we are at least able to conjecture the form of the optimal decision with some confidence, that is

CONJECTURE. *If $s_2 < r$, and s_3 is large, the optimal decision is one of $(0, s_2, r - s_2)$, $(0,0,r)$, $(1, s_2, r - s_2 - 1)$, $(1, 0, r-1)$, depending on F and q.*

That is, we either choose to reclassify companies from C_3 immediately, or look to minimise immediate cost and reclassify companies from C_3 with the remaining companies we are allowed to invite to bid. Unfortunately attempts to prove this conjecture have proved fruitless. Of course once $s_2 \geq r$ we no longer have a control problem and the optimal decision will clearly be one of $(0, r, 0)$ or $(1, r-1, 0)$ depending on F.

Obviously this simple model needs to be made more general to be deemed practical. A first step is to assume F and q are unknown and allow ourselves to update estimates of these at each time period, incorporating these estimates in the state vector.

A further generalisation is to allow companies in C_1 and C_2 to drift back into C_3. So we would need to continually reclassify companies by inviting them to bid. We would then also need to be updating estimates of these drift probabilities.

The original aim of this thesis was to develop a game model for the whole problem based on the ideas of Harsanyi (1967,1968a,b) and latterly Aumann (1987). The first stage in this process is to build simple decision theoretic models for the bidders and tenderer. This process has taken virtually the whole time available and as a result there has been no time to work on the game theoretic model, but briefly the approach is as follows.

Define the players in the game to be the tenderer and the n bidders. Then presumably this will be a game of *incomplete information*, i.e. a given player will be unaware of his competitors attributes, for example the utilities they will be using. Nearly all existing game theory litereature discusses games with complete information, concentrating on developing solution concepts. Harsanyi is one of the few to have discussed games with incomplete information. The main difficulty with the analysis of these games is the *infinite regress*. For example in a two player game where the competitors do not know each others utilities it is natural for player 1 to make his move conditional on what he expects player 2's utility to be. But he must also make his move conditional on what he expects player 2's expectation of his utility will be, and so on ad infinitum. Of course in a many player game this regress becomes even more convoluted.

Harsanyi claims to have overcome this problem by defining a game with complete information (G^*) that is game theoretically equivalent to the game with incomplete information (G). A solution to the game G^* can then be given without encountering the problems of the infinite regress. Briefly the idea is as follows. Each player has associated with him an *attribute vector*, **a**, consisting of the unknowns about that player, for example his utility. It is then assumed that the components of **a** are randomly selected from some known distribution prior to the players deciding on their moves. Thus when the players decide their moves it will be a game of complete, but *imperfect*, information, this will be the game G^*. A game of imperfect information is one in which players do not have the full history of previous moves or chance events. Solution concepts for these games have been studied extensively.

The main problem with the Harsanyi ideas, which are very well respected, is that there are virtually no practical examples of a reduction from a game G to a game G^*. So the building of a Harsanyi model for a complex game such as that arising in our problem would be a substantial and significant achievement.

Of course a number of the more general areas are always ripe for further research. In particular the development of a general multivariate dynamic generalised linear model would be very desirable. Some of the problems that would be encountered in attempting to do this are illustrated in the development of the model in §4.4, for example the problems with constraints.

Another area is that of the GMABP. The aim here would be to prove that some form of index policy is optimal. Identifying the form of this index would not then be too difficult. An appropriate approach may be similar to that used by Nash (1980) and discussed in the review of §7.2.

Finally one must mention what is the least developed area of theory discussed in this thesis, namely that of finitely additive distributions. The A_n distributions and the o.i. distributions of chapter 6 are just about the only finitely additive distributions to be shown to have a practical application in decision theory. It seems very likely that there are many problems where other classes of finitely additive distributions could prove very useful.

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